Symbolic-Numeric Computation
D. Wang and L. Zhi, Eds.
Trends in Mathematics, 349–359

A Symbolic-Numeric Approach to an Electric Field Problem

David J. Jeffrey, Silvana Ilie, James M. Gardiner and Steven W. Campbell

Abstract. A combination of symbolic and numerical methods is used to extend the reach of the purely symbolic methods of physics. One particular physics problem is solved in detail, namely, a computation of the electric potential in the space between a sphere and a containing cylinder. The potential is represented as an infinite sum of multipoles, whose coefficients satisfy an infinite system of linear equations. The system is solved first symbolically by using a series expansion in a critical ratio, namely the ratio of the sphere radius to cylinder radius. Purely symbolic methods, however, cannot complete the solution for two reasons. First, the coefficients in the series expansion can only be found numerically, and, second, the convergence rate of the series is too slow. The combination of symbolic and numerical methods allows the singular nature of an important special case to be identified.

Mathematics Subject Classification (2000). Primary 35C99; Secondary 68W25; Tertiary 65B99.

Keywords. Laplace equation, series solution, asymptotic solution, convergence, numerical analysis, symbolic analysis.

1. Introduction

Many problems in theoretical physics are solved using purely symbolic methods. Such problems, however, are usually restricted to simple situations, for example, a sphere falling slowly through an infinite fluid, or flow past a two-dimensional airfoil. In contrast, symbolic methods have little success with more realistic problems. For example, consider the problem of trying to calculate the flow of a fluid around an object in a tube. The traditional symbolic methods of fluid mechanics were applied by Happel and Brenner [2], who made a lengthy symbolic, but approximate, calculation for a small sphere inside a much larger tube. The problem of a sphere of a reasonable size cannot be solved accurately by their methods. In addition, their calculation was restricted to particles of spherical shape. With these observations in mind, supporters of purely numerical methods argue that symbolic methods are not capable of contributing efficiently to the solution of flow problems like these. This implies that computer systems such as Maple, which facilitate symbolic manipulation, are equally incapable of contributing to flow problems. Further, given progress in the automatic generation of computational grids and computational schemes for numerical methods, there is a danger that symbolic methods will be largely pushed out of many areas of physics and engineering, and remain present only in the initial set up of a problem.

In response to this situation, we present here a symbolic–numeric approach to one class of physics problems, which we describe with the aid of a specific problem, namely the electric field around a non-conducting spherical bubble in a cylindrical wire, a problem first considered in [1]. The starting point is an expansion in eigenfunctions, a classical method of 19th century physics. By itself this fails because the coefficients cannot be found in closed form. Therefore one tries to compute them numerically. This is still not very successful because the series converge very slowly and further analysis is needed. By combining a numerical calculation of the coefficients with a symbolic analysis of the series, we arrive at a useful expression for the resistance of the wire.

The new method offers advantages both when compared with purely numerical methods, and when compared with purely symbolic methods. On the one hand, when compared with purely symbolic methods, the present methods get a solution, which otherwise is beyond reach. On the other hand, when compared with numerical solutions, the present solution retains symbolic information, and moreover very useful information. Specifically, there is a ratio of lengths in the specification of the problem, namely the ratio of the diameter of the spherical bubble to the diameter of the tube, and this ratio is present as a symbolic parameter, which means that the solution obtained is valid for all ratios, whereas a numerical solution requires a complete repetition of the solution procedure. A further advantage of the symbolic–numeric approach is the fact that the problem contains a singular limit. When the bubble nearly fills the tube, i.e., when the ratio of diameters approaches 1, there can be singular effects. The presence of a symbolic parameter in the solution allows the singular behaviour to be studied analytically.

Another aspect of the problem should be noted. The information required from the solution to the problem has an important influence on the solution technique. Here we are interested in the effect of the bubble on the resistance of the wire. Thus we need to calculate one quantity, namely the additional resistance or equivalently the increased effective length. It turns out that this quantity can be extracted neatly from the solution. If we had been interested in something different, for example, the details of the electric field around the bubble, the current method may be of less interest. It is one of a number of techniques being developed to extend the role of symbolic computation in Science and Engineering.

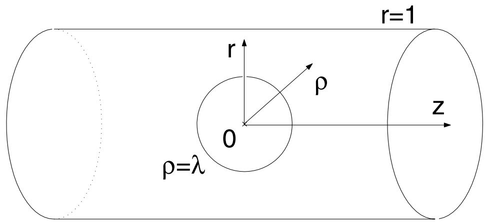

FIGURE 1. The coordinate systems for the sphere inside the cylinder.

Laplace's equation described the electric potential associated with an electric field, and has been studied in the space outside a sphere and inside a cylinder by Linton [5]. Linton used the method of multipoles to derive a solution of the problem. His technique consists of first constructing a set of functions that satisfy the equation and all the boundary conditions except the one on the sphere, and then representing the solution as a superposition of all the functions in the set. Satisfying the condition on the sphere leads to an infinite set of linear equations in the coefficients of the solution. The new feature of the present paper consists in expressing the unknown coefficients as series expansions in the geometrical parameter referred to above: the ratio of the sphere diameter to the cylinder diameter. The solution is exact for all diameter ratios, but its rate of convergence slows down in the limit of the particle blocking the tube.

When the particle nearly blocks the tube, a different approach can be used. An asymptotic analysis allows us to solve the problem approximately. By comparing the general solution and the asymptotic solution, we predict the behaviour of the coefficients in the general solution, and thereby improve its rate of convergence.

2. Solution for All Diameter Ratios

We consider the electric field present in a cylindrical tube of radius d containing a sphere of radius a situated on the axis. The field is produced by a potential gradient, which causes a current to flow through the tube; equivalently, the electric field tends to a constant at infinity. We use the cylindrical coordinate system (r, z, ϕ) and the spherical coordinate system (ρ, θ, ϕ) which both have their origins at the centre of the sphere and are rescaled so that the cylinder boundary corresponds to $r = 1$. The spherical boundary is then given by $\rho = \lambda$ where λ is the ratio of the sphere radius to the cylinder radius. The axial symmetry is used to suppress reference to the azimuthal angle ϕ. The two systems of coordinates are connected by the relations $z = \rho \cos\theta,\ r = \rho \sin\theta$.

We introduce an electric potential of the form $z + \Phi$. Because of the linearity of Laplace's equation, we can scale the potential so that the electric field tends to unity at infinity. The disturbance potential Φ must satisfy Laplace's equation in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2 \Phi}{\partial z^2}=0\ ,$$

together with the boundary conditions

$$\frac{\partial \Phi}{\partial r} = 0\ , \qquad \text{on}\quad r=1\ , \tag{2.1}$$

$$\frac{\partial \Phi}{\partial \rho} = -\cos\theta\ , \qquad \text{on}\quad \rho=\lambda\ , \tag{2.2}$$

$$\frac{\partial \Phi}{\partial z} \to 0\ , \quad \text{as}\quad |z|\to\infty\ . \tag{2.3}$$

The last equation is not equivalent to $\Phi \to 0$ as might be expected, but rather to

$$\Phi \to \alpha \operatorname{sgn} z\ , \quad \text{as}\quad |z|\to\infty\ , \tag{2.4}$$

where α depends on λ and is in general non-zero [4]. The consequences of this for the convergence of the series used below are discussed in [4].

To solve the problem above we use dual expansions. First we express the solution in cylindrical coordinates as a linear combination of functions satisfying the equation and the boundary conditions (2.1) and (2.4). Then we transform the solution into spherical coordinates and apply the condition (2.2).

Starting in cylindrical coordinates, we expand the perturbed potential as a series by using the usual separation of variables. Thus assuming $\Phi = R(r)Z(z)$ (the azimuthal angle ϕ does not enter because of axisymmetry), we find that $R(r)$ is a linear combination of Bessel functions $K_0(tr)$ and $I_0(tr)$, where t^2 is the separation constant. Similarly $Z(z)$ is a combination of $\sin tz$ and $\cos tz$. To satisfy boundary condition (2.1), we must combine the Bessel functions according to $K_0(tr) + [K_1(t)/I_1(t)]I_0(tr)$. Symmetry in z requires that the $\cos tz$ term is dropped. Then integrating over all values of t, we obtain

$$\Phi = \sum_{n=1}^{\infty}\frac{A_n}{2n}\Phi_n \tag{2.5}$$

where

$$\Phi_n = \frac{2\lambda^{2n+1}(-1)^{n+1}}{\pi(2n-1)!}\int_0^\infty t^{2n-1}\left(K_0(tr)+\frac{K_1(t)}{I_1(t)}I_0(tr)\right)\sin(tz)dt\ . \tag{2.6}$$

The following identities allow the transformation between the potential in cylindrical coordinates (r,z) and in spherical coordinates (ρ,θ):

$$\frac{1}{\rho^{2n}}P_{2n-1}(\cos\theta) = \frac{2(-1)^{n+1}}{\pi(2n-1)!}\int_0^\infty t^{2n-1}K_0(tr)\sin(tz)dt\ ,$$

$$I_0(tr)\sin(tz) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)!}(t\rho)^{2n-1}P_{2n-1}(\cos\theta)\ .$$

By applying these identities we deduce

$$
\begin{aligned}
\Phi &= \sum_{n=1}^{\infty} \frac{A_n}{2n} \frac{\lambda^{2n+1}}{\rho^{2n}} P_{2n-1}(\cos\theta) \\
&+ \sum_{m,n=1}^{\infty} \frac{2A_n(-1)^{n+m}}{\pi(2n)!(2m-1)!} \lambda^{2n+1} \rho^{2m-1} P_{2m-1}(\cos\theta) \int_0^{\infty} t^{2(n+m-1)} \frac{K_1(t)}{I_1(t)} dt \ .
\end{aligned}
$$

From the boundary condition (2.2) and the orthogonality of the Legendre functions we obtain an infinite system of algebraic equations for A_n

$$
A_n + \sum_{m=1}^{\infty} \lambda^{2n+2m-1} A_m B_{mn} = \delta_{1n} \qquad (n = 1, 2, \ldots) \tag{2.7}
$$

with the coefficients B_{mn} given by

$$
\begin{aligned}
B_{mn} &= \frac{2(-1)^{n+m+1}}{\pi(2m)!(2n-2)!} \int_0^{\infty} t^{2(n+m-1)} \frac{K_1(t)}{I_1(t)} dt \\
&= \frac{2(-1)^{n+m+1}}{\pi(2m)!(2n-2)!(2n+2m-1)} \int_0^{\infty} \frac{t^{2(n+m-1)}}{I_1^2(t)} dt \ .
\end{aligned}
$$

This set of equations was obtained in [1] before computer algebra was readily available. However, the derivation was checked using Maple, but a completely automatic program was not written. To solve the equations, one could truncate them and tackle them numerically at this point, but to do so would be to return to a purely numerical solution, and then it would be dubious whether all of the manipulations above were worthwhile, or whether a finite-element scheme would not be just as good. Therefore, we solve the system of equations by expressing each coefficient as a series in λ; this choice is suggested by the appearance of the term $\lambda^{2n+2m-1}$ in (2.7).

$$
A_n(\lambda) = \sum_{s=0}^{\infty} K_{ns} \lambda^s \ . \tag{2.8}
$$

Substituting this into (2.7) and collecting powers of λ, we obtain a recurrence relation for the K_{np} coefficients:

$$
K_{np} + \sum_{s=1}^{(p+1-2n)/2} K_{s(p+1-2s-2n)} B_{sn} = 0 \ , \quad \text{for } \ n \geq 1, \quad p \geq 2n+1 \ , \tag{2.9}
$$

and

$$
\begin{aligned}
K_{n0} &= \delta_{n1} \ , \quad \text{for } \ n \geq 1 \\
K_{np} &= 0 \ , \quad \text{for } \ n \geq 1, \ 2n \geq p \geq 1.
\end{aligned}
$$

The numerical solution of (2.9) for the numbers K_{np} is straightforward. Notice that although the problem contains the parameter λ, these are pure numbers independent of λ. In order to speed up the computation and to improve its reliability, we can integrate the dominant contribution to the integral symbolically. Thus, we write

$$\int_0^\infty \frac{t^{2(n+m-1)}}{I_1^2(t)}\,dt = \frac{\pi(2n-1)!}{2^{2n-1}} + \int_0^\infty t^{2(n+m-1)}\left[I_1^{-2}(t) - 2\pi t e^{-2t}\right]\,dt\ .$$

It can be remarked that supporters of computer algebra frequently say that their systems are not purely symbolic, but numeric as well. This aspect of MAPLE made the calculation of the coefficients very easy because the integrals could be evaluated easily using MAPLE's numerical integration.

It was stated in the introduction that the solution, which now has been presented, is attractive because we want only the effective resistance to an electric current flowing past the bubble. According to (2.4), the difference in potential increases by the amount 2α, which is *a priori* unknown. By considering the asymptotic behaviour of the integral in (2.6), we find that for $n = 1$ it is asymptotic to $\operatorname{sgn} z$ and for $n > 1$ it tends to 0 (details in [4]). Therefore α depends only on A_1. The full analysis gives

$$\Delta\Phi = 2\alpha = 2\lambda^3 A_1\ . \tag{2.10}$$

Thus the resistance depends only on the single coefficient A_1, and therefore will be expressed as a single series with computable coefficients. The above solution has the advantage that a single computation of the system (2.9) solves the problem for all λ.

An apparent disadvantage is the fact that the series diverges at $\lambda = 1$, which corresponds to the sphere filling the tube. For values of λ near the radius of convergence, the rate of convergence of (2.8) becomes very slow and many terms is needed in the series. However, this effect has a physical basis: the problem is singular when the sphere fills the tube, and even purely numerical methods struggle in this case. This apparent disadvantage, however, can be turned into an advantage in the context of symbolic-numeric computation. We can now analyze the singular limit and match the limit to the general solution. The result is a new series that converges everywhere numerically and which displays the singular behaviour symbolically. This is only possible in a symbolic-numeric context.

Asymptotic Behaviour

We want to study the asymptotic nature of the solution when the sphere is almost the same diameter as the cylinder. The gap between the sphere and the cylinder is measured by the non-dimensional parameter $\varepsilon = 1 - \lambda$ which is assumed much smaller than 1.

We proceed by considering a stretching transformation of the cylindrical coordinates in the gap, based on the physical fact that the effects across the small gap dominate the effects along it. A similar idea in rather different geometry [3] suggests that the proper transformation is

$$R = (1-r)/\varepsilon, \quad Z = z/\sqrt{\varepsilon}.$$

The surface of the sphere in the new coordinates has the expansion

$$R = 1 + \frac{1}{2}Z^2 + \varepsilon\left(\frac{Z^2}{2} + \frac{Z^4}{8}\right) + O(\varepsilon^2).$$

We will apply the boundary condition on this approximation rather than on the exact surface, since the exact expression contains square roots.

The scaled Laplace's equation inside the gap is given by

$$\frac{\partial^2\Phi}{\partial R^2} + \varepsilon\left(\frac{\partial^2\Phi}{\partial Z^2} - \frac{1}{1-\varepsilon R}\cdot\frac{\partial\Phi}{\partial R}\right) = 0 \tag{2.11}$$

and the boundary condition on the cylinder by

$$\frac{\partial\Phi}{\partial R} = 0 \qquad\qquad \text{on} \quad R = 0.$$

In order to deduce the boundary condition on the obstacle we notice that

$$\sin\theta = \frac{1-\varepsilon R}{1-\varepsilon}, \qquad\qquad \cos\theta = \frac{\sqrt{\varepsilon}Z}{1-\varepsilon} \qquad \text{on} \quad \rho = 1-\varepsilon$$

and derive the scaled boundary condition on the sphere

$$\frac{\partial\Phi}{\partial R} - \varepsilon\left(R\frac{\partial\Phi}{\partial R} + Z\frac{\partial\Phi}{\partial Z}\right) = \varepsilon^{3/2}Z. \tag{2.12}$$

We apply Gauss' theorem between $z \to -\infty$ and $z = 0$ and find that $\Phi = O(\varepsilon^{-1/2})$ in the gap. Consequently, it is natural to look for an expansion of the potential of the form

$$\Phi(R, Z) = \varepsilon^{-\frac{1}{2}}\Phi_0(R, Z) + \varepsilon^{\frac{1}{2}}\Phi_1(R, Z) + \cdots.$$

The unknown functions Φ_n are derived by replacing the above expansion in the problem (2.11)-(2.12) and by matching the expansions inside and outside the gap. The first approximation leads to the following problem for Φ_0

$$\begin{aligned}
\frac{\partial^2\Phi_0}{\partial R^2} &= 0, \\
\frac{\partial\Phi_0}{\partial R} &= 0 \qquad \text{on} \quad R = 0, \\
\frac{\partial\Phi_0}{\partial R} &= 0 \qquad \text{on} \quad R = 1 + \frac{Z^2}{2},
\end{aligned}$$

which has the solution $\Phi_0 = K(Z)$. The next step in the approximation gives

$$\begin{aligned}
\frac{\partial^2\Phi_1}{\partial R^2} &= -K''(Z), \\
\frac{\partial\Phi_1}{\partial R} &= 0 \qquad\qquad \text{on} \quad R = 0, \\
\frac{\partial\Phi_1}{\partial R} &= ZK'(Z) \qquad \text{on} \quad R = 1 + \frac{Z^2}{2}.
\end{aligned} \tag{2.13}$$

From (2.13) we find that K must satisfy

$$(1+\frac{1}{2}Z^2)K''+ZK'=0$$

and using the antisymmetry with respect to Z, we derive $\Phi_0 = C\cdot\arctan(Z/\sqrt{2})$. The constant C should be obtained from a matching with the solution outside the gap, but that would require deriving a solution in the 'outer' region beyond the gap. We can avoid this by using Gauss' theorem to determine the constant C:

$$\Phi = \frac{1}{\sqrt{2\varepsilon}}\cdot\arctan\frac{Z}{\sqrt{2}}+O(\varepsilon^{1/2}). \tag{2.14}$$

Matching General Solution with Asymptotic Solution

The asymptotic solution (2.14) behaves asymptotically like $\operatorname{sgn} z$ with respect to z. Therefore, from (2.10), we obtain

$$A_1 = \frac{\pi}{2\sqrt{2}}(1-\lambda)^{-1/2}+O((1-\lambda)^{1/2}).$$

Thus, if we expand the asymptotic solution with respect to λ, we obtain the series

$$A_1 = \frac{\pi}{2\sqrt{2}}\sum_{p=1}^{\infty}(-1)^p\binom{-1/2}{p}\lambda^p+O((1-\lambda)^{1/2}).$$

We also made the assumption for the general solution that the coefficient expands as (2.8). By matching the two forms of the solution we get the prediction

$$\frac{K_{1p}}{\frac{\pi}{2\sqrt{2}}(-1)^p\binom{-1/2}{p}}\to 1 \qquad \text{as} \quad p\to\infty\ .$$

This prediction is tested in Fig. 2, where it can be seen that the agreement is very good after about 50 terms. A selection of the same data expressed in tabular form is given in Table 1.

The application that suggested this calculation is the change in the electrical resistance of a wire owing to impurities in the metal, and for this only the coefficient A_1 is of interest. The extra resistance is often expressed as an effective increase in the length of the cylinder, and this extra length is given by

$$\Delta L = 2\lambda^3 A_1.$$

Since we know the singular behaviour symbolically, we can extract it from the numerical coefficients and obtain a more reliable calculation. We write

$$\begin{aligned} A_1 &= \sum_{n=0}^{\infty}K_{1n}\lambda^n && (2.15)\\ &= \frac{\pi}{\sqrt{8}(1-\lambda)^{1/2}}+\sum_{n=0}^{\infty}\left[K_{1n}-\frac{\pi}{\sqrt{8}}(-1)^n\binom{-1/2}{n}\right]\lambda^n. && (2.16)\end{aligned}$$

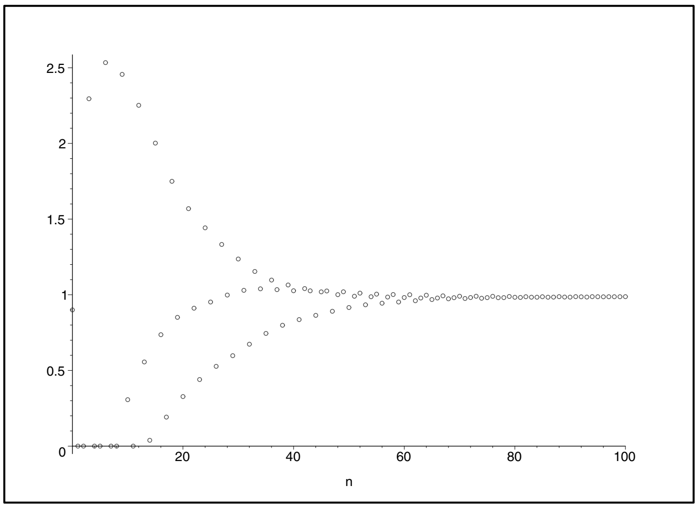

FIGURE 2. The ratio of the coefficients K_{1n} as computed from (2.9) to the predicted value $\frac{\pi}{2\sqrt{2}}(-1)^n\binom{-1/2}{n}$, for different values of n.

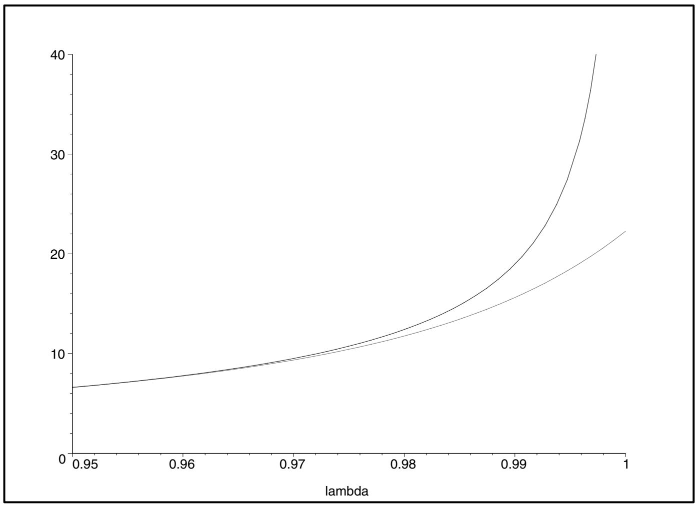

FIGURE 3. The computed resistance increase as a function of λ for values close to 1. The upper curve shows the correct singular behaviour which is expressed symbolically; the lower curve shows the purely numerical result.

TABLE 1. A comparison of the computed and predicted coefficients in the series for the increased resistance of a conducting cylinder due to the presence of a spherical bubble.

n	K_{1n}	$(-1)^n\pi\binom{-1/2}{n}/\sqrt{8}$	Difference
95	0.063370	0.064209	-0.000839
96	0.062993	0.063876	-0.000883
97	0.062742	0.063547	-0.000805
98	0.062402	0.063222	-0.000820
99	0.062086	0.062903	-0.000817
100	0.061827	0.062587	-0.000760

In Table 1, the last column shows how much smaller the new difference coefficients are, and Fig. 3 shows the effect of extracting the singularity explicitly on the results.

3. Conclusions

This paper has shown that by combining symbolic and numeric techniques, we can obtain new forms for the solution of problems arising in physics. The method used here can be extended to other equations of theoretical physics, such as Stokes's equations and Helmholtz's equation. The principles, although probably not the detailed method, can be extended to other geometries. In general, purely numerical methods will remain more flexible than the present one, but what has been demonstrated is that when symbolic information can be returned to a solution, there is a gain in numerical accuracy and in our understanding of the solution. One pleasing feature of the current method is that the two solutions used were derived independently, and hence the agreement between the two ways of computing the coefficients is a good check on the correctness of the intermediate working.

References

[1] J. M. Gardiner, *A contained potential problem, with special reference to the resistivity of cavitated α-brass*, M.Sc. thesis, University of Melbourne, 1971.

[2] J. Happel, H. Brenner, *Low Reynolds Number Hydrodynamics*, Prantice-Hall, Englewood Cliffs, 1965.

[3] D. J. Jeffrey, *The temperature field or electric potential around two almost touching spheres*, J. Inst. Math. Appl., vol. 22, pp. 337–351, 1978.

[4] S. Ilie, D. J. Jeffrey, *A note on Laplace's equation inside a cylinder*, Appl. Math. Lett., vol. 18 (1), pp. 55–59, 2005.

[5] C. M. Linton, *Multipole methods for boundary-value problems involving a sphere in a tube*, IMA J. Appl. Math., vol. 55, pp. 187–204, 1995.

[6] W. R. Smythe, *Flow around a spheroid in a circular tube*, Phys. Fluids, vol. 7, pp. 633–7, 1964.

[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

David J. Jeffrey, Silvana Ilie, James M. Gardiner and Steven W. Campbell
Department of Applied Mathematics
University of Western Ontario
London, Ontario, Canada N6A 5B7

!!! Please tell us your email-address