The asymptotic analysis of some interpolated nonlinear recurrence relations

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ABSTRACT

We study discrete dynamical systems, or recurrence relations, of the general form

$$y_{n+1} = y_n\phi(y_n) = y_n(1 + \alpha_1 y_n + \alpha_2 y_n^2 + \cdots)$$

with explicitly known series coefficients α_k and $\alpha_1 \neq 0$. We associate with the discrete system an interpolating continuous system $Y(t)$, such that $Y(n) = y_n$. The asymptotic behaviour of y_n can then be investigated through $Y(t)$. The corresponding continuous system is

$$Y'(t) = G(Y(t)) \ , \tag{1}$$

where G is called the generator (following Labelle's terminology), and is given by an explicit formula in terms of the recurrence relation. This continuous system may fail to be smooth everywhere but nonetheless may be useful. Analytic solution is only rarely possible.

We analyze the equation for Y under assumptions of an asymptotic limit, and show that the asymptotic behaviour can be obtained by reverting a series containing logarithms and powers. We introduce a novel reversion based on the Wright ω function.

An application of the theory is made to functional iteration of the Lambert W function and the asymptotic behaviour of the iteration is obtained.

The iteration of functions is a central topic in the theory of complex dynamical system, and a sophisticated use of conjugation is only one key tool used there. We show here that Labelle's theory and generator can be used to compute the conjugated mapping of functional iterations to simple non-iterative functions in general. We use the Lambert W function again as an example to illustrate this. We also discuss the curious asymptotic series $\ln z \sim \sum_{k\geq 1} W^{<k>}(z)$.

This study uses the truncated generalized series tools available in Maple, particularly the logarithmic-and-power series that is usual in Maple. We also use Levin's u-transform as a key piece in interpolating the discrete dynamical system.

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Categories and Subject Descriptors

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General Terms

Algorithms, Theory

1. INTRODUCTION

Interpolation is of great value in many branches of mathematics. Thus, the Γ function interpolates the factorial function, fractional-order derivatives interpolate ordinary derivatives [1] and the method of modified equations interpolates the numerical solution of differential equations [5], to name a few examples. The last named (modified equations) is susceptible to treatment by power series in a computer-algebra environment [6], although, of course, one would prefer analytic solution if possible. One lovely example is described in [13], where the iteration $x_{n+1} = (x_n - 1/x_n)/2$ is interpolated to get $x_t = \cot 2^t\theta_0$, where $\theta_0 = \cot x_0$. This interpolation sheds much light on the chaotic nature of Newton's method.

In this paper, we show how to combine formal power series methods, using the theory of Labelle [9], with Levin's u-transform and numerical integration to achieve some of the same ends. The specific context is that of discrete dynamical systems, and more specifically, the asymptotic analysis of nonlinear recurrence relations. The aforementioned work by Labelle [9] developed a very interesting algebra of formal power series under composition; we use the ideas to study the embedding of certain discrete dynamical systems into interpolating continuous dynamical systems. This is valuable because the asymptotic behaviour of a continuous dynamical system is sometimes easier to understand than the behaviour of a nonlinear recurrence relation. The asymptotic analysis of nonlinear recurrence relations demands a wide variety of tools, and the present approach is one such useful tool.

We study discrete dynamical systems, or recurrence relations, of the general form

$$y_{n+1} = F(y_n) = y_n\phi(y_n) = y_n(1 + \alpha_1 y_n + \alpha_2 y_n^2 + \cdots) \tag{2}$$

with explicitly known series coefficients α_k and $\alpha_1 \neq 0$. This form is slightly more general and more interesting than it first looks. It has a neutrally stable fixed point at $y = 0$, with $F'(0) = 1$. A dynamical system with a fixed point at $y = \mu$ can be cast in this form by translation.

The fixed point is asymptotically ambiguous with a multiplier $F'(0) = 1$; the generic cases $|F'(0)| < 1$, for which the fixed point is attractive, and $|F'(0) > 1|$, for which it is repelling, are more common but easier to understand.

We associate with the discrete system an interpolating continuous system $Y(t)$, such that $Y(n) = y_n$. The asymptotic behaviour of y_n can then be investigated through $Y(t)$. A theorem of Labelle states that the corresponding continuous system is

$$Y'(t) = G(Y(t)) \ , \qquad (3)$$

where G is termed the generator, and is given by

$$G(y) = \frac{\alpha_1 y^2}{1 + \beta_1 y + \beta_2 y^2 + \cdots} \ . \qquad (4)$$

The β_k are related to the α_k by the recurrence relation

$$\beta_0 = 1, \qquad (5)$$

$$\beta_n = \frac{1}{n\alpha_1} \sum_{i=1}^{n} \left[P_{i+1}(3) - \frac{(n+3)P_{i+1}(n+2-i)}{n+2-i} \right] \beta_{n-i}. \qquad (6)$$

where $P_n(s)$ is the coefficient of y^n in $\phi^s(y)$, that is,

$$\begin{aligned} \phi^s(y) &= (1 + \alpha_1 y + \alpha_2 y^2 + \cdots)^s \\ &= 1 + P_1(s)y + \cdots + P_k(s)y^k + \cdots \ . \end{aligned} \qquad (7)$$

This is an extension of Lagrange inversion, as is now well-known.

As an introductory example, we consider a system with a closed form solution:

$$y_{n+1} = y_n/(1 + y_n) = y_n(1 - y_n + y_n^2 \dots) \ . \qquad (8)$$

This has the solution

$$y_n = y_0/(1 + ny_0) \ , \qquad (9)$$

which can be verified by induction.

We now disregard this known solution and show how Labelle's theorem finds a continuous function $Y(t)$ which interpolates y_n. Comparing (8) with the definitions just given, we see $\alpha_k = (-1)^k$, and therefore we have

$$P_n(s) = \binom{-s}{n} = \frac{(-1)^n s^{\overline{n}}}{n!} \ . \qquad (10)$$

Here we use the standard 'n rising' notation. Therefore the P_{i+1} appearing in (6) become

$$P_{n+1}(3) = \frac{(-1)^{n+1}(n+3)(n+2)}{2} \ , \qquad (11)$$

$$P_{n+1}(2) = (-1)^{n+1}(n+2) \ . \qquad (12)$$

When $i = n$ in (6), we have $P_{n+1}(3) - \frac{n+3}{2} P_{n+1}(2) = 0$. Hence by induction we can prove that $\beta_k = 0$ for $k = 1, 2, \cdots$, and therefore the generator in this example is

$$G(Y) = -Y^2 \ , \qquad (13)$$

and so $Y' = -Y^2$ with the solution

$$Y(t) = \frac{y_0}{1 + ty_0} \ . \qquad (14)$$

Thus $Y(n) = y_n$ as required, and the asymptotic behaviour is easily verified to be $y_n \sim 1/n$, as long as $y_0 > 0$.

In the above example, the $\beta_k = 0$, and we need a more general example. Our second example obeys the recurrence relation

$$y_{n+1} = W(y_n) = y_n - y_n^2 + \frac{3}{2}y_n^3 - \frac{8}{3}y_n^4 + \cdots \qquad (15)$$

where W is the principal branch of the Lambert W function [4]. Applying again Labelle's theorem, we find the coefficients for the new generator $G(Y)$, whose general form is given in (4). We have $\alpha_1 = -1$ and

$$\beta_1 = \frac{1}{2},\ \beta_2 = -\frac{1}{6},\ \beta_3 = \frac{1}{8},\ \beta_4 = -\frac{19}{180},\ \beta_5 = \frac{1}{12}\ , \dots \ , \qquad (16)$$

where we note specifically that $\beta_1 \neq 0$.

Returning to the general case, we rewrite (3)–(4) in the form

$$\frac{1 + \beta_1 Y + \beta_2 Y^2 + \cdots}{\alpha_1 Y^2} \frac{dY}{dt} = 1. \qquad (17)$$

This separable equation can be integrated term by term to give

$$-\frac{1}{Y} + \beta_1 \ln Y + \beta_2 Y + O(Y^2) = \alpha_1 t + C. \qquad (18)$$

2. AN ACCURATE SERIES REVERSION

So far, we have done little more than recapitulate Labelle's work. What we find interesting is that assuming $y_n \to 0+$ implies $Y(t) \to 0$ as $t \to \infty$. Then the generator equation can be used to find the detailed asymptotic behaviour of y_n by using series inversion (also called reversion) of the log-and-power series.

We assume $\beta_1 \neq 0$ in (18), or there is no logarithmic term. Typically, this series will be divergent, so identification of the constant of integration C from the numerical initial condition $Y(t = t_0) = y_0$ will require some form of sequence acceleration, such as Levin's u-transformation. We assume for the moment that this has been done and that C can be considered known.

If $Y \to 0$ as $t \to \infty$, then the two dominant terms are $-1/Y$ and $\beta_1 \ln Y$. Instead of using a power-and-then-log expansion directly, we make use of the fact that a zeroth order approximation (call it Z) may be found by solving the *two*-term first approximation:

$$-\frac{1}{Z} + \beta_1 \ln Z = \alpha_1 t + C \ .$$

Once this has been identified, we look for a series expansion of the form

$$Y \sim Z + c_2 Z^2 + c_3 Z^3 + \ldots = Z + \sum_{k \geq 2} c_k Z^k \ , \text{ as } Z \to 0 \ .$$

Although there is a well-developed multivariate generalization of Lagrange inversion which might be used here, our purpose is well enough served by the simple approach of substituting the series into the defining equation, expanding and equating coefficients. The expansion of the log term is made simpler by the introduction of the following function.

For $z \in \mathbb{C}$, the unwinding number is defined by [2, 8] as

$$\mathcal{K}(z) = \left\lceil \frac{\Im z - \pi}{2\pi} \right\rceil \ , \qquad (19)$$

with $\Im z$ being the imaginary part of z. We then have the property

$$z = \ln(e^z) + 2\pi i \mathcal{K}(z) \ . \qquad (20)$$

The unwinding number is needed for the correct manipulation of logarithms of complex argument, through identities such as:

$$\ln z_1 z_2 = \ln z_1 + \ln z_2 - 2\pi i \mathcal{K}(\ln z_1 + \ln z_2) \ . \qquad (21)$$

More such identities can be found in [2].

The expansion of logarithm is thus

$$\ln Y \sim \ln\left(Z + \sum_{k\geq 2} c_k Z^k\right) \quad (22)$$

$$\sim \ln Z + \ln\left(1 + \sum_{k\geq 2} c_k Z^{k-1}\right) - 2\pi i \mathcal{K}_0 \quad (23)$$

$$\sim \ln Z + \sum_{k\geq 1} d_k Z^k - 2\pi i \mathcal{K}_0 \ , \quad (24)$$

where $\mathcal{K}_0 = \mathcal{K}(\ln Z + \ln(1 + \sum_{k\geq 2} c_k Z^{k-1}))$. The d_k are easily computed functions of the c_k by well-known algorithms for the computation of the logarithm of a series [5]. Similarly, by reciprocation of a series, we can obtain

$$\frac{1}{Y} \sim \frac{1}{Z}\left(1 + \sum_{k\geq 1} e_k Z^k\right) \ ,$$

with $e_1 = -c_2$ and this allows easy grouping of terms. We obtain

$$-\frac{1}{Z} - \sum_{k\geq 1} e_k Z^{k-1} + \beta_1(\ln Z + \sum_{k\geq 1} d_k Z^k - 2\pi i \mathcal{K}_0)$$
$$+\beta_2\left(Z + c_2 Z^2 + c_3 Z^3 \ldots\right)$$
$$+\beta_3\left(Z + c_2 Z^2 + c_3 Z^3 \ldots\right)^2 + \ldots = \alpha_1 t + C \ .$$

From this we see that $-e_1 - 2\pi i \mathcal{K}_0 \beta_1 = c_2 - 2\pi i \mathcal{K}_0 \beta_1 = 0$. By considering the limit $Z \to 0$, we see that $\mathcal{K}_0 = 0$, using the fact that $\mathcal{K}(\ln Z) = 0$, thus implying $Y \sim Z + c_3 Z^3 + \ldots$.

In order to obtain a closed-form expression for Z, we use the Wright ω function which is defined in terms of the branches of Lambert W.

$$\omega(z) = W_{\mathcal{K}(z)}(e^z) \ . \quad (25)$$

The equation $y + \ln y = z$ has a solution in terms of Wright ω.

$$y = \begin{cases} \text{No solution} \ , & \text{for } z = t - i\pi \text{ and } t \leq -1 \\ \omega(z) \ , \omega(z - 2\pi i) \ , & \text{for } z = t + i\pi \text{ and } t \leq -1 \\ \omega(z) & \text{otherwise} \ . \end{cases} \quad (26)$$

To apply the above to the solution of

$$-\frac{1}{Z} + \beta_1 \ln Z = \alpha_1 t + C \ ,$$

we again must invoke the unwinding number to ensure correctness, since Z might be complex. Since analysis using $\mathcal{K}$ is still not standard, we give several of the intermediate steps. Assuming $\beta_1 \neq 0$ (the interesting case)

$$-\frac{1}{\beta_1 Z} + \ln Z + \ln(\beta_1) = \frac{\alpha_1 t + C}{\beta_1} + \ln(\beta_1),$$
$$-\frac{1}{\beta_1 Z} + \ln(\beta_1 Z) + 2\pi i \mathcal{K}(\ln Z + \ln(\beta_1))$$
$$= \frac{\alpha_1 t + C}{\beta_1} + \ln(\beta_1),$$
$$-\frac{1}{\beta_1 Z} + \ln(\frac{1}{1/(\beta_1 Z)}) = \frac{\alpha_1 t + C}{\beta_1} + \ln(\beta_1) - 2\pi i \mathcal{K}_1,$$
$$-\frac{1}{\beta_1 Z} - \ln(\frac{1}{\beta_1 Z}) - 2\pi i \mathcal{K}(-\ln(\frac{1}{\beta_1 Z}))$$
$$= \frac{\alpha_1 t + C}{\beta_1} + \ln(\beta_1) - 2\pi i \mathcal{K}_1.$$

We now have a form to which the solution (26) applies, and therefore

$$\frac{1}{\beta_1 Z} = \omega\left(-\frac{\alpha_1 t + C}{\beta_1} - \ln(\beta_1) + 2\pi i(\mathcal{K}_1 - \mathcal{K}_2)\right) \ ,$$

where $\mathcal{K}_1 = \mathcal{K}(\ln Z + \ln(-\beta_1))$ and $\mathcal{K}_2 = \mathcal{K}(-\ln(\frac{1}{\beta_1 Z}))$. Normally both $\mathcal{K}_1$ and $\mathcal{K}_2$ will be zero.

Since $\omega(z) \sim z - \ln z + \sum_{n\geq 0}\sum_{m\geq 1} c_{nm} \frac{\ln^m z}{z^{n+m}}$, outside the rays of discontinuity we have

$$\begin{aligned} Y \ &= \ -\frac{1}{\alpha_1 t} + \frac{C + \ln(\beta_1)\beta_1}{\alpha_1{}^2 t^2} \\ &- \ \frac{C^2 + 2C\ln(\beta_1)\beta_1 + (\ln(\beta_1))^2 \beta_1{}^2 + \beta_2}{\alpha_1{}^3 t^3} \\ &+ \ \frac{C^3 + 3C^2\ln(\beta_1)\beta_1 + 3C(\ln(\beta_1))^2\beta_1{}^2}{\alpha_1{}^4 t^4} \\ &+ \ \frac{(\ln(\beta_1))^3\beta_1{}^3 + \beta_1\beta_2 + \beta_3 + 3\beta_2 C + 3\beta_2\ln(\beta_1)\beta_1}{\alpha_1{}^4 t^4} \\ &+ \ \widetilde{O}\left(t^{-5}\right) \end{aligned} \quad (27)$$

This can be done using Maple commands `Wrightomega` and `asympt` [10, 12].

Here the symbol $\widetilde{O}$ is called soft-oh notation [14]. This notation is used as a variant of big-oh notation that ignores logarithmic factors. For example, $f(n) = \widetilde{O}(g(n))$ is shorthand for $f(n) = O(g(n)\log^k g(n))$ for some k.

The details of higher order approximation will involve the β_k which depend on the α_k, and will be most neatly expressed in terms of powers of this function.

We observe that although the above calculations were presented in the context of example 2 equation (15), we have made no use of the numerical coefficients (16) other than $\beta_1 \neq 0$. Therefore, we can state the following theorem.

THEOREM 1. *If $y_{n+1} = y_n(1 + \alpha_1 y_n + \alpha_2 y_n^2 + \cdots)$, then $Y(t)$ interpolating y_n with the initial condition y_0 given satisfies*

$$\frac{(1 + \beta_1 Y + \beta_2 Y^2 + \cdots)}{\alpha_1 Y^2}\frac{dY}{dt} = 1 \quad (28)$$

or

$$-\frac{1}{Y} + \beta_1 \ln Y + \beta_2 Y + \cdots = \alpha_1 t + C \quad (29)$$

where

$$C = -\frac{1}{Y_0} + \beta_1 \ln Y_0 + \beta_2 Y_0 + \frac{\beta_3}{2} Y_0^2 + \cdots \ . \quad (30)$$

This series is divergent, but can be evaluated by Levin's u-transform [11], which is implemented in Maple as well.

Here $Y_0 = y_0$ and moreover, if $\beta_1 \neq 0$

$$Y(t) \sim Z + c_3 Z^3 + c_4 Z^4 + \cdots \text{ as } t \to \infty \tag{31}$$

where

$$Z = \frac{1}{\beta_1 \omega(-\frac{\alpha_1 t + C}{\beta_1} - \ln(\beta_1) + 2\pi i(\mathcal{K}_1 - \mathcal{K}_2))}, \tag{32}$$

and the c_k's are given by

$$c_3 = \beta_2,\ c_4 = \beta_1\beta_2 + \beta_3,\ c_5 = \beta_1^2\beta_2 + \beta_1\beta_3 + 2\beta_2^2 + \beta_4, \cdots$$

The proof is a repetition of the calculations already presented, apart from the computation of the c_k. These follow naturally on insertion of the putative series for Y into the equation for Y, namely equation (18), and equating coefficients.

Here we invert (29) using the Wright ω function. One may ask why not just invert equation (29) directly to obtain an asymptotic expression of $Y(t)$ as $t \to \infty$? The answer is that we obtain more information from the asymptotic series using the Wright ω function than inverting it directly. The first two terms of (31) are $Z + c_3 Z^3$ without a term in z^2. On the contrary, if we invert it directly, we need another term to get the same information and the cost to compute the coefficients is higher. Therefore, our approach is superior when compared with the traditional approach.

3. COMPUTING THE SUMS BY LEVIN'S U-TRANSFORMATION

Levin's sequence transformation is designed to be exact for model sequences of the following type:

$$s_n = s + \omega_n \sum_{j=0}^{k-1} c_j/(n+\beta)^j \ . \tag{33}$$

Recall from (6) that the coefficients of the generator in equation (30) are given by the following recurrence:

$$\beta_0 = 1, \tag{34}$$

$$\beta_n = \frac{1}{n\alpha_1}\sum_{i=1}^{n}\left[P_{i+1}(3) - \frac{(n+3)P_{i+1}(n+2-i)}{n+2-i}\right]\beta_{n-i}. \tag{35}$$

where $P_n(s)$ is the coefficient of y^n in $\phi^s(y)$ in equation (7).

The partial sums are

$$s_n = \sum_{i=2}^{n} \beta_i Y_0^{i-1}. \tag{36}$$

The partial sum s_n can be written as follows:

Using equation (35),

$$s_n = \sum_{j=2}^{n-1}\beta_j Y_0^{j-1} + \beta_n Y_0^{n-1} \tag{37}$$

$$= \sum_{j=2}^{n-1}\beta_j Y_0^{j-1} + \sum_{i=1}^{n} f(n,i)\beta_{n-i} Y_0^{n-1} \tag{38}$$

where $f(n,i) = \frac{1}{n\alpha_1}[P_{i+1}(3) - \frac{(n+3)P_{i+1}(n+2-i)}{n+2-i}]$.

In our example 2 where the Lambert W function is considered, $P_n(s) = -\frac{s^n}{n!}$. So when $i = 1$, $P_{i+1}(n+2-i) = P_2(n+1) \sim O(n^2)$ which means $f(n,1) \sim O(n)$. Similarly, $f(n,2) \sim O(n^2), f(n,3) \sim O(n^3)$. We can assume $f(n,i) \sim O(n^i)$ when i is small. Therefore we see that β_n is growing rapidly and we can safely drop the small terms and have $\beta_n \sim \sum_{i=1}^{\sigma} O(n^i)\beta_{n-i}$ where σ is a small number. Inductively, we may assume $\beta_n \sim O(n)\cdot\beta_{n-1}$ when n is large. Therefore $s_n \sim \beta_n Y_0^{n-1}$. Our computation also confirms this.

Although this argument above is for our specific example, in general, we observe by computations that when n is large and i is small, for instance, $i = 1, \cdots, 10$, $f(n,i)$ is large and the second sum on the right of equation (37) is dominant over the first sum.

We choose $\omega_n = \beta_n Y_0^{n-1}$. Our computation shows that the ratio $\omega_n/s_n = 1 + O(1/n)$ as $n \to \infty$. Therefore, the remainder $r_n = s_n - s$ has an asymptotic expansion:

$$r_n = s_n - s = \omega_n(c_0 + c_1/n + O(1/n^2)). \tag{39}$$

"The Levin transformation should work very well for a given sequence $\{s_n\}$ if the sequence $\{\omega\}$ of remainder estimates is chosen in such a way that ω_n is proportional to the dominant term of an asymptotic expansion of the remainder r_n" [15] (7.3-1 in Page 42).

4. NUMERIC VERIFICATION OF REVERSION

We now use our example 2, given in equation (15) to verify numerically the reversion proposed above. We use the notation $W^{<n>}$ for the nth iterate of W. We take $z = 0.1$ and consider the difference between $W^{<100>}(z)$ and the predictions of the asymptotic series to $O(Z^4)$.

The asymptotic series of Y is

$$Y = Z + \frac{1}{6}Z^3 - \frac{7}{48}Z^4 + O(Z^5)\ . \tag{40}$$

For $t = 100$ and $Y_0 = 0.1$, we compute $C = -11.16736741$ by equation (30), which gives $Z = 0.009189275412$. Then from (40), we obtain

$$\widetilde{Y} = Z + \frac{1}{6}Z^3 - \frac{7}{48}Z^4 = 0.009189403700\ . \tag{41}$$

All computations were carried out in 100-digit precision. By Maple, the accurate result is $W^{<100>}(z) = 0.009189403721$.

The relative error is thus

$$\eta = \frac{\widetilde{Y} - W^{<100>}(y_0)}{W^{<100>}(y_0)} = 1.15 \times 10^{-9}\ . \tag{42}$$

The predicted relative error, using the first omitted term from (40), is

$$\widetilde{\eta} \approx \frac{c_5 Z^5}{Z} = \frac{707}{4320}Z^4 = 1.16 \times 10^{-9}. \tag{43}$$

The predicted relative error is close to the actual relative error, meaning the asymptotic series (40) is a good approximation of the iterations of the Lambert W function.

We also compare the reversion to the traditional direct approach, which reverts (29) using only powers and logarithms. Our starting point becomes the approximation

$$\frac{-1}{Y} = \alpha_1 t + C\ ,$$

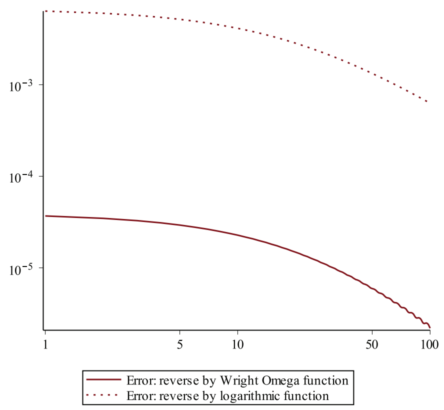

Figure 1: A log-log plot of the errors using equations (40) and (44). The horizontal axis is the continuous variable t.

to obtain $Y \approx Y_0 = -1/(\alpha_1 t + C)$. We recall $\alpha_1 < 0$ so $Y_0 > 0$. For $t \gg 1$, we shall have $Y_0 \ll 1$, and a small parameter for an expansion. We therefore seek a series solution of equation (29) in the form $Y = Y_0(1+\delta)$ where

$$\begin{aligned} \delta &= -\beta_1 Y_0 \ln(Y_0) + a_1 Y_0 + a_2 Y_0^2 \ln^2(Y_0) \\ &\quad + a_3 Y_0^2 \ln Y_0 + a_4 Y_0^2 + \widetilde{O}(Y_0^3) \ . \end{aligned}$$

Using Maple, we extend this to include $O(Y_0^3)$ terms and obtain the series

$$\begin{aligned} \widetilde{Yt} &= Y_0 - 1/2\, Y_0^2 \ln Y_0 + 1/4\, Y_0^3 \ln^2 Y_0 \\ &+ 1/4\, Y_0^3 \ln Y_0 + 1/6\, Y_0^3 \ . \end{aligned} \tag{44}$$

Evaluation at $t = 100$ gives $\widetilde{Yt}(100) = 0.009189352211$ and the error is 5.6×10^{-6}. In Fig. 1, we compare the two approaches to reversion for a range of values of t.

Remark. We have found that if the initial z is not small enough, Levin's u-transformation will fail to identify the constant of integration C. Therefore, for large z, we first apply enough iterations of F (here W) to get into the range of initial conditions that allow identification of C.

5. DYNAMICS OF LAMBERT W

We now return to the motivation for our example, the iterations of W itself. We illustrate the dynamics of the discrete dynamical system $W^{<t>}(y_0)$ interpolated by the generator equation (28) in Fig. 2.

In Fig. 2, we plot the iterates of four different initial values $z = Y_0$. We use the initial points $z = Y_0 = \{3 \pm 3i, 0.8 + i, 0.4 + i\}$. In each case, the iteration $W^{<t>}(Y_0)$ approaches the origin as $t \to \infty$.

The iterates of W were initially interesting to us because of the following result. Starting from the definition of the principal branch $W_0(z)$, and assuming $z > 0$, we note

$$W_0(z) e^{W_0(z)} = z > 0$$

implies $W_0(z) > 0$ and hence

$$\log W_0(z) + W_0(z) = \log z \ ,$$

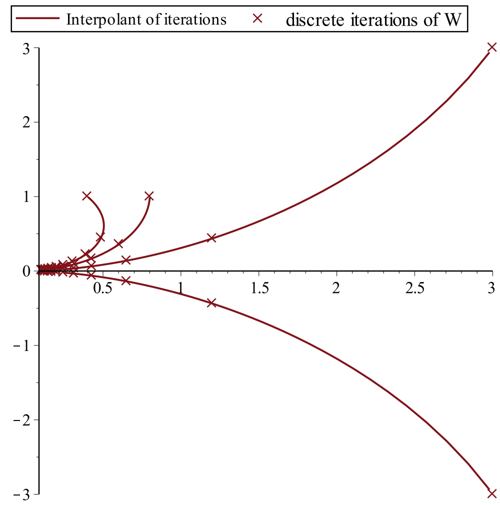

Figure 2: Dynamics of $W^{<t>}(y_0)$. The iterations start at $W^{<0>} = z$, where $z = 3 \pm i, 0.8 + i, 0.4 + i$. The circles mark the discrete interates, while the solid lines show the continuous interpolant.

or, dropping the branch indicator 0,

$$\begin{aligned} \log z &= W(z) + \log W(z) \\ &= W(z) + W(W(z)) + \log W(W(z)) \\ &= W(z) + W(W(z)) + W(W(W(z))) \\ &+ \log W(W(W(z))) \\ &= \sum_{k=1}^{N} W^{<k>}(z) + \log W^{<N>}(z), \end{aligned}$$

so long as $W^{<k>}(z) > 0$ for $0 \le k \le N$. This gives the curious asymptotic relationship

$$\log z \sim \sum_{k \ge 1} W^{<k>}(z) \tag{45}$$

which inverts the well-known asymptotic relation

$$\begin{aligned} W(z) &\sim \log z - \log\log(z) \\ &+ \sum_{n\ge 0}\sum_{m \ge 1} c_{nm} \frac{(\log\log(z))^m}{(\log z)^{n+m}} \quad \text{as } z \to \infty \end{aligned} \tag{46}$$

where $c_{nm} = \frac{1}{m!}(-1)^n \left[{n+m \atop n+1} \right]$ is expressed in terms of Stirling cycle numbers.

In view of the above results, we have the following theorem for the iterations $W^{<k>}(z)$.

Theorem 2. *For any $z \in \mathbb{C}$, $W^{<k>}(z) \to 0$ as $k \to \infty$.*

Proof. The range of $W(z) = W_0(z)$ is contained in the semi-infinite rectangle $-1 \le \Re(z) < \infty$, $-\pi \le \Im(z) \le \pi$. Moreover $\Re(W(z)) = O(\log|z|)$ and hence the image of any bounded disk $|z| < R$ is contained wholly within the intersection of the rectangle and the disk, but since $\log R < R$, this is inside the disk. Hence by the fixed point theorem [7] (page 524) $W^{<k>}(z)$ tends to a fixed point; but there is only the fixed point $z = 0$. See Figure 3 to illustrate this argument. □

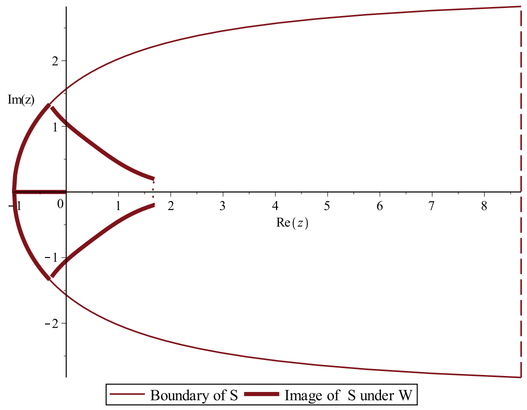

Figure 3: The region contained within the thin solid line is denoted S and consists of the range of the principal branch W_0. The images of the boundary points of S under W_0 are shown as heavy lines and illustrate that the image of S is contained in S. Thus W_0 is a contraction mapping.

COROLLARY 1. *The series* $\sum_{k\geq 1} W^{<k>}(z)$ *diverges.*

PROOF. $\ln z = \sum_{k=1}^{N} W^{<k>}(z) + \ln W^{<N>}(z)$ if $z > 0$ and since $W^{<N>}(z) \to 0$, the remainder term $\ln W^{<N>}(z) \to -\infty$ as $N \to \infty$. $\square$

Remark. It may seem counterproductive to replace $\ln z$ with an asymptotic series that (a) contains only more complicated functions than $\ln z$ and (b) diverges anyway, but since for fixed N, $W^{<N>}(z) \to \infty$ as $z \to \infty$, we have that $\ln z \sim \sum_{k\geq 1} W^{<k>}(z)$ as $z \to \infty$, and the simplicity of the series means that it ought to be easy to replace $\ln z$ in asymptotic applications with an appropriate finite expansion in terms of iterates of W, and this may simplify some analyses.

6. CONJUGATION

We study the iteration of $f(z)$ using the method of conjugation mapping from complex dynamics [3]. The notation for iterations of $f(z)$ is defined by $f^{<n>}(z) = f(f^{<n-1>}(z))$ with $f^0(z) = z$. Given a function $f(z)$, by conjugation with a function $\phi(z)$ we mean replacing $f(z)$ with $z \to (\phi \circ f \circ \phi^{-1})(z)$. Then iterates of f may be more easily studied if the conjugated map is simpler. In this section, we provide an apparently new computation of a valuable conjugation function (in series) by using the series approach developed in previous sections.

Suppose $f(z) = z + az^2 + \cdots$. By conjugating f by $C(z) = -az$, we can show that without loss of generality, we may assume $a = -1$.

We now consider conjugating $f(z)$ by the inversion mapping $\phi : z \to \frac{1}{z}$, that is $g(z) = \phi \circ f \circ \phi^{-1}$. Then $g(z)$ has the following form:

$$g(z) = z + 1 + \frac{b}{z} + \cdots \tag{47}$$

Now define

$$\varphi_n(z) = g^{<n>}(z) - n - b \log n, \tag{48}$$

we have

$$\varphi_{n+1}(z) = g^{<n+1>}(z) - n - 1 - b \log(n+1) \tag{49}$$

and

$$\varphi_n(g(z)) = g^{<n>}(g(z)) - n - b \log n. \tag{50}$$

By combining the two equations together, we have

$$\varphi_n(g(z)) = \varphi_{n+1}(z) + 1 + b \log(1 + \frac{1}{n}) \tag{51}$$

From Lennart [3], we have $\varphi_n \to \varphi$ uniformly as $n \to \infty$. Therefore, if we let $n \to \infty$ in (51), then $b \log(1 + \frac{1}{n}) \to 0$ and we have

$$\varphi(g(z)) = \varphi(z) + 1\ . \tag{52}$$

That is $\varphi \circ g = \varphi + 1$ and we have

$$\varphi \circ g \circ \varphi^{-1} : z \to z + 1\ . \tag{53}$$

Again conjugated by inversion mapping $\phi : z \to \frac{1}{z}$, we have

$$\frac{1}{z} \circ \varphi \circ g \circ \varphi^{-1} \circ \frac{1}{z} = \frac{z}{1+z}\ . \tag{54}$$

The iteration of (54) has a simple form, that is:

$$\begin{aligned}(\frac{1}{z} \circ \varphi \circ g \circ \varphi^{-1} \circ \frac{1}{z})^{<n>} &= \frac{1}{z} \circ \varphi \circ g^{<n>} \circ \varphi^{-1} \circ \frac{1}{z} \\ &= \frac{z}{1+nz}\end{aligned} \tag{55}$$

Since $g(z) = \phi \circ f \circ \phi^{-1} = \frac{1}{z} \circ f \circ \frac{1}{z}$, we have the following

$$\Phi \circ f^{<n>} \circ \Phi^{-1} = z \to \frac{z}{1+nz}, \tag{56}$$

where $\Phi = \frac{1}{z} \circ \varphi \circ \frac{1}{z}$.

Since $\varphi_n \to \varphi$, if we let $\Phi_n = \frac{1}{z} \circ \varphi_n \circ \frac{1}{z}$, we will have $\Phi_n \to \Phi = \frac{1}{z} \circ \varphi \circ \frac{1}{z}$. This gives us the idea of how to compute Φ.

Remember $\varphi_n(z) = g^{<n>}(z) - n - b \log n$ from (48), we proceed as follows:

$$\begin{aligned}\Phi_n &= \frac{1}{z} \circ \varphi_n \circ \frac{1}{z} && (57)\\ &= \frac{1}{z} \circ (g^{<n>}(z) - n - b \log n) \circ \frac{1}{z} && (58)\\ &= \frac{1}{g^{<n>}(\frac{1}{z}) - n - b \log n} && (59)\\ &= \frac{1}{\frac{1}{z} \circ f^{<n>}(z) \circ \frac{1}{z} \circ \frac{1}{z} - n - b \log n} && (60)\\ &= \frac{1}{\frac{1}{f^{<n>}(z)} - n - b \log n} && (61)\\ &= \frac{f^{<n>}(z)}{1 - (n + b \log n) f^{<n>}(z)} && (62)\\ & && (63)\end{aligned}$$

We may compute it directly, but then it involves computation of the iteration $f^{<n>}(z)$, which has some disadvantages. Alternatively, we use the theory above to seek another approach.

Recall equation (31), we have

$$f^{<t>}(z) \sim Z + c_3 Z^3 + c_4 Z^4 + \cdots \tag{64}$$

as $t \to \infty$.

We ignore the higher order term and substitute $f^{<t>}(z) = Z$ into (62) and use the Maple command *asympt* to calculate the asymptotic series of Φ_n. Surprisingly, we get

$$\Phi_n(z) = -\frac{1}{C(z)} + O(\frac{1}{n}) \tag{65}$$

That means

$$\Phi(z) = \lim_{n\to\infty} \Phi_n(z) = -\frac{1}{C(z)} \tag{66}$$

This gives a new approach for computing $\Phi(z)$.

We use the Lambert W function as an example to compare this new approach to the straightforward approach. We call the straightforward approach method 1 and our new approach method 2. We compute $\Phi(0.1)$ by each method.

Method 1:

- n = 100, $\Phi(0.1) = 0.08989916406$
- n = 1000, $\Phi(0.1) = 0.08957875912$

Method 2:

- n = 5, $\Phi(0.1) = 0.08954662020$
- n = 20, $\Phi(0.1) = 0.08954661947$
- n = 50, $\Phi(0.1) = 0.08954661947$

As we can see, method 2 has the advantage of converging faster than method 1.

7. CONCLUDING REMARKS

It has long been known in the analysis of algorithms community that using the Lambert W function (also called sometimes the Omega function, prior to its name being standardized) was useful in economizing some asymptotic analyses. This present paper shows an example class of problems for which the economization is made explicit: instead of a log-and-power expansion for Y, we have $Y \sim Z+O(Z^3)$, gaining an order because $c_2 = 0$.

This paper also presents the use of the Wright ω function for this purpose; and gives some results on the iteration of W itself, motivated by the series expansion (45). Note that using (31) one may formally rewrite any series containing logarithm into one containing W; this may also be of interest.

Finally, we have presented an apparently new method of computing a useful conjugation map that transports the dynamics of the iterates of f to a standard, easily understood function.

8. ACKNOWLEDGMENT

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