PCH PhysicoChemical Hydrodynamics Vol. 10, No. 4, pp. 461–470, 1988.
Printed in Great Britain

0191-9059/88 \$3.00+0.00
Pergamon Press plc

FORCES AND STRESSLETS FOR THE AXISYMMETRIC MOTION OF NEARLY TOUCHING UNEQUAL SPHERES

D. J. JEFFREY and R. M. CORLESS

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada

(*Received* 13 *April* 1988)

Abstract—The hydrodynamic interactions between two unequal spheres immersed in a low Reynolds number flow are calculated. Explicit expressions for the force and the stresslet that each sphere exerts on the fluid are derived, assuming that the spheres are nearly touching and that their motions are axisymmetric. The results are asymptotic expansions based on the width of the gap between the spheres, which must be small when compared with either of the two radii. The force and stresslet results give the asymptotic behaviour of the resistance functions which describe the axisymmetric low Reynolds number interactions between a pair of spheres. Two specific problems are considered. In the first, one sphere approaches the other along the line of centres and in the second, one sphere deforms axisymmetrically and uniformly. It is shown that the new asymptotic results agree with numerical data that have been published for these functions and moreover, the asymptotic results can be used to improve the convergence of series expressions for the resistance functions.

1. INTRODUCTION

The axisymmetric approach of two unequal spheres has been studied by Cooley and O'Neill (1969), Hansford (1970) and Jeffrey (1982) with the aim of calculating asymptotically the forces exerted by the spheres on the fluid. These forces are not the only quantities of interest, however, because the analysis of the properties of suspensions of particles requires also the stresslets of the spheres, defined by Batchelor (1970) as:

$$S = -\int_A [\tfrac{1}{2}(\mathbf{x}'\boldsymbol{\sigma}\cdot\mathbf{n}+\boldsymbol{\sigma}\cdot\mathbf{n}\mathbf{x}')-\tfrac{1}{3}\mathbf{I}\mathbf{x}'\cdot\boldsymbol{\sigma}\cdot\mathbf{n}]\,\mathrm{d}A. \qquad (1.1)$$

The vector $\mathbf{x}'$ is drawn from the centre of the sphere. Various authors have defined resistance functions to describe the forces and stresslets exerted by each of 2 spheres on the suspending fluid (Brenner and O'Neill, 1972; Schmitz and Felderhof, 1982; Jeffrey and Onishi, 1984b; Kim and Mifflin, 1985). Using the notation of Jeffrey and Onishi (1984b) and Kim and Mifflin (1985), we can reduce the resistance functions that describe force and stresslet interactions in axisymmetric flows to the scalar functions $X^A_{\alpha\beta}$, $X^G_{\alpha\beta}$ and $X^M_{\alpha\beta}$ (defined below for $\alpha, \beta = 1, 2$). Of these, only the $X^A_{\alpha\beta}$ functions have been investigated with any thoroughness. For them, we have asymptotic results for the cases of widely separated spheres and nearly touching ones and numerical data for most other parts of the domain of the functions (Jeffrey and Onishi, 1984b). In contrast, $X^G_{\alpha\beta}$ and $X^M_{\alpha\beta}$ are known only through numerical data available for equal spheres (Kim and Mifflin, 1985). This paper provides asymptotic results for $X^G_{\alpha\beta}$ and $X^M_{\alpha\beta}$, which are valid when the separation between the spheres is small. In order to obtain these results, we found it convenient to formulate the calculations differently from earlier papers. Previous calculations worked only with the stream function ψ, because the force on a sphere in an axisymmetric flow can be expressed entirely in terms of ψ. This is not so for the stresslet, however. For its calculation we require the pressure and the velocity field, so if one has to obtain explicit expressions for velocity and pressure it is just as convenient to work from the start in these variables.

The new approach is also of interest because it is much closer in spirit and detail to the calculations made in non-axisymmetric flows by, for example, O'Neill and Stewartson (1967) and Corless and Jeffrey (1988). The main difference between the axisymmetric and non-axisymmetric cases is that the Reynolds equation for the pressure is simpler in the axisymmetric case and whereas O'Neill and Stewartson could obtain only asymptotic estimates for the homogeneous solutions to the equation, here we can obtain closed form solutions. Apart from this, the principles of the earlier calculations carry over and will not be re-iterated in detail. Thus, the fluid motion in the gap is analysed and the flow outside the gap is not. This does not mean that the flow outside the gap is not important, but rather it means we can avoid calculating it if we restrict our attention to the singular terms in the forces and stresslets (O'Neill and Stewartson, 1967). To calculate the non-singular terms we must calculate the solution outside the gap; we do not do that here and instead use existing results to calculate the required terms. One feature of the earlier calculations that is taken a step further here is the use of a computer to perform the algebra. Previously, the CAMAL system was used (Jeffrey, 1982); for this work the Maple system was used. The newer language has made several steps easier (not to say possible); for example, the solution of the Reynolds equation in closed form.

The rationale for studying deforming spheres needs a little explanation. The usual definitions of the resistance matrix for Stokes flow consider rigid spheres translating and rotating in a shear flow (Jeffrey and Onishi, 1984b; Kim and Mifflin, 1985). The linearity of the Stokes equations allows us to express the forces F_α, couples L_α and stresslets S_α exerted by sphere α (a = 1, 2) as:

$$\begin{pmatrix} F_1 \\ F_2 \\ L_1 \\ L_2 \\ S_1 \\ S_2 \end{pmatrix} = \mu \begin{pmatrix} A_{11} & A_{12} & \tilde{B}_{11} & \tilde{B}_{12} & \tilde{G}_{11} & \tilde{G}_{12} \\ A_{21} & A_{22} & \tilde{B}_{21} & \tilde{B}_{22} & \tilde{G}_{21} & \tilde{G}_{22} \\ B_{11} & B_{12} & C_{11} & C_{12} & \tilde{H}_{11} & \tilde{H}_{12} \\ B_{21} & B_{22} & C_{21} & C_{22} & \tilde{H}_{21} & \tilde{H}_{22} \\ G_{11} & G_{12} & H_{11} & H_{12} & M_{11} & M_{12} \\ G_{21} & G_{22} & H_{21} & H_{22} & M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} U_1 - U(x_1) \\ U_2 - U(x_2) \\ \Omega_1 - \Omega \\ \Omega_2 - \Omega \\ -\mathscr{E} \\ -\mathscr{E} \end{pmatrix}. \tag{1.2}$$

In this equation, U_α is the velocity of the centre of sphere α and $U(x_\alpha)$ is the ambient velocity at x_α the position of the sphere's centre; Ω_α is the angular velocity of sphere α and Ω is the ambient angular velocity; $\mathscr{E}$ is the ambient rate of strain. Once S_1 has been calculated, it is clear that M_{11} and M_{12} will be known only in the combination $M_{11}+M_{12}$. However, it is desirable to separate the M functions because numerical evidence given in Jeffrey (1988) clearly shows that the asymptotic behaviour of each scalar function within M is different and the numerical evaluation of the functions benefits from recognizing this. Therefore, we should like to formulate a problem that will lead to separate estimates of M_{11} and M_{12}. Even if these problems are artificial, the mathematical benefits are worthwhile. With this aim, we suppose that each sphere can have a uniform rate of strain $\mathscr{E}_\alpha$. The no-slip condition continues to apply even to these deforming spheres. Then (1.2) remains valid, with the replacement of the terms $-\mathscr{E}$ by rates of strain $\mathscr{E}_1 - \mathscr{E}$ and $\mathscr{E}_2 - \mathscr{E}$. By selecting the particular axisymmetric rates of strain:

$$\mathscr{E}_1 = \mathscr{E}(\mathbf{kk} - \tfrac{1}{3}\mathbf{I}) \quad \text{and} \quad \mathscr{E}_2 = 0,$$

where $\mathbf{k}$ is a unit vector along the line of centres of the spheres, together with the conditions

$U_\alpha = \Omega_\alpha = \mathscr{E} = 0$, we can reduce the stresslets to expressions containing the scalar functions X^M_{11} and X^M_{21}:

$$S_1 = M_{11} : \mathscr{E}_1 = X^M_{11}(\mathbf{kk} - \tfrac{1}{3}\mathbf{I}) \quad \text{and} \quad S_2 = M_{21} : \mathscr{E}_1 = X^M_{21}(\mathbf{kk} - \tfrac{1}{3}\mathbf{I}).$$

Thus, from S_1 and S_2 we can now obtain separate asymptotic estimates of X^M_{11} and X^M_{21}, to improve on the purely numerical separation achieved in Jeffrey (1988).

The governing equations that apply to both flow problems are as follows. We suppose the spheres have radii a and b and that their centres are on the z axis at $z = a+h$ and $z = -b$. The gap width is clearly h. The non-dimensional gap width is $\varepsilon = h/a$ with $\varepsilon \ll 1$. We introduce non-dimensionalized, stretched cylindrical coordinates by the relations:

$$Z = z/a\varepsilon \quad \text{and} \quad R = r/a\varepsilon^{1/2}.$$

If $\mathscr{V}$ is a velocity scale, as yet unspecified, we can write the velocities as $\mathscr{V}(u, 0, w)$ and the pressure as $\mu\mathscr{V}p/a$, where μ is the viscosity. The velocity and pressure are then scaled and expanded in powers of ε as follows:

$$u = \varepsilon^{-1/2}U = \varepsilon^{-1/2}[U_1 + \varepsilon U_2 + \mathrm{O}(\varepsilon^2)], \tag{1.3}$$

$$w = W = W_1 + \varepsilon W_2 + \mathrm{O}(\varepsilon^2), \tag{1.4}$$

$$p = \varepsilon^{-2}P = \varepsilon^{-2}[P_1 + \varepsilon P_2 + \mathrm{O}(\varepsilon^2)]. \tag{1.5}$$

After making the usual approximations of low Reynolds number flow, the scaled governing equations are the continuity equation:

$$\partial(RU)/\partial R + R\,\partial W/\partial Z = 0. \tag{1.6}$$

and the Stokes equations:

$$\partial P/\partial Z = \varepsilon\nabla^2 W \quad \text{and} \quad \partial P/\partial R = \nabla^2 U - \varepsilon U/R^2. \tag{1.7}$$

The ∇^2 operator has also been scaled and here is given by:

$$\nabla^2 = \frac{\partial^2}{\partial Z^2} + \varepsilon\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}. \tag{1.8}$$

The surfaces of the spheres in the neighbourhood of the gap can be expanded in powers of ε. For the first sphere (radius a) we have:

$$Z = 1 + \tfrac{1}{2}R^2 + \varepsilon\tfrac{1}{8}R^4 + \mathrm{O}(\varepsilon^2) = H_1 + \varepsilon\tfrac{1}{8}R^4 + \mathrm{O}(\varepsilon^2)$$

and for the second sphere:

$$Z = -(a/2b)R^2 - \varepsilon(a/2b)^3R^4 + \mathrm{O}(\varepsilon^2) = \tfrac{1}{2}\kappa R^2 + \varepsilon\tfrac{1}{8}\kappa^3R^4 + \mathrm{O}(\varepsilon^2).$$

The κ is the same as that defined in Jeffrey (1982). In addition we shall use H_2 for $\kappa R^2/2$.

2. APPROACHING SPHERES

We shall suppose that the sphere of radius a approaches the other, which is at rest, at velocity U. The scale velocity $\mathscr{V}$ is then U and the boundary conditions to be applied on sphere a are:

$$U = 0 \quad \text{and} \quad W = -1 \quad \text{on } Z = H_1 + \varepsilon\tfrac{1}{8}R^4,$$

The boundary conditions on sphere b are:

$$U = 0, \quad W = 0, \quad \text{on } Z = H_2 + \varepsilon\tfrac{1}{8}\kappa^3 R^4.$$

Since:

$$U(R, Z = H_1 + \varepsilon\tfrac{1}{8}R^4) = U_1(R, H_1) + \varepsilon U_2(R, H_1) + \varepsilon\tfrac{1}{8}R^4[\partial U_1/\partial Z]_{Z=H_1},$$

we can equate powers of ε to find:

$$U_1(R, H_1) = 0, \quad U_2(R, H_1) = -\tfrac{1}{8}R^4[\partial U_1/\partial Z]_{Z=H_1}.$$

In a similar way we find:

$$W_1(R, H_1) = -1, \quad W_2(R, H_1) = -\tfrac{1}{8}R^4[\partial W_1/\partial Z]_{Z=H_1}.$$

On sphere b we have:

$$U_1(R, H_2) = 0, \quad U_2(R, H_2) = -\tfrac{1}{8}\kappa^3 R^4[\partial U_1/\partial Z]_{Z=H_2}.$$

$$W_1(R, H_2) = 0, \quad W_2(R, H_2) = -\tfrac{1}{8}\kappa^3 R^4[\partial W_1/\partial Z]_{Z=H_2}.$$

Substituting (1.3)–(1.5) into (1.6)–(1.8), we obtain equations for P_1, U_1 and W_1 by equating powers of ε.

$$\partial P_1/\partial Z = 0, \quad \partial P_1/\partial R = \partial^2 U_1/\partial Z^2, \quad \partial(RU_1)/\partial R + R\,\partial W_1/\partial Z = 0.$$

Integrating the first two equations with respect to Z, we obtain:

$$P_1 = P_1(R), \quad U_1 = \tfrac{1}{2}P_1'(Z - H_1)(Z - H_2), \tag{2.1}$$

where $P_1' = \mathrm{d}P_1(R)/\mathrm{d}R$. From the last equation we obtain:

$$W_1 = -\frac{1}{12R}(RP_1')'(2Z - 3H_1 + H_2)(Z - H_2)^2 + \frac{1}{4}RP_1'[(1-\kappa)(Z - H_2) + 2\kappa(Z - H_1)](Z - H_2), \tag{2.2}$$

where the condition $W_1(Z = H_2) = 0$ has been applied. The final boundary condition $W_1(Z = H_1) = -1$ will be satisfied only if:

$$(RP_1')' + 3(1-\kappa)R^2 P_1'/(H_1 - H_2) + 12R/(H_1 - H_2)^3 = 0. \tag{2.3}$$

This is the Reynolds equation for axisymmetric flow. In non-axisymmetric flows the pressure P_1 appears explicitly as a separate term, making the equation impossible to solve completely. Here the solution is:

$$P_1 = 3/[(1-\kappa)(H_1 - H_2)^2] + \mathrm{C}[(H_1 - H_2)^{-1} + \tfrac{1}{2}(H_1 - H_2)^{-2} + \ln R^2 - \ln(H_1 - H_2)] + \mathrm{K},$$

where C and K are constants of integration. Since the solution must remain bounded at $R = 0$, we must have $\mathrm{C} = 0$. Since P_1 must decay to zero as we leave the gap ($R \to \infty$) we must have $K = 0$. Substituting the pressure into (2.1) and (2.2) completes the solution to the first order; it agrees with that found previously using the stream function.

The solution at the next order is similar. The equations are:

$$\partial P_2/\partial Z = \partial^2 W_1/\partial Z^2, \quad \partial P_2/\partial R = \frac{\partial^2}{\partial Z^2}U_2 + \frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}U_1 - U_1/R^2,$$

$$\partial(RU_2)/\partial R + R\,\partial W_2/\partial Z = 0.$$

The obvious new feature is the presence of known functions of R and Z coming from the first order solution. One finds that the new functions and the different boundary conditions affect only the inhomogeneous terms in the differential equation for P_2 and so the integration of the equivalent to (2.3) is only more difficult through the inhomogeneous terms being longer. We shall not present any more intermediate steps, however, because the manipulations were performed using Maple, an algebra manipulation language from the University of Waterloo. Even our own record of the calculations is nothing more than a file of Maple commands. The Reynolds equation was solved by a standard Maple routine, something the previously used CAMAL system (Jeffrey, 1982; Jeffrey and Onishi, 1984a) could not do.

The asymptotic form of the pressure near the edge of the gap ($R \to \infty$) is important at each order because it must match with the (not derived) pressure outside the gap and because the singular terms in the resistance functions, which it is our object to find, depend on the asymptotic behaviour. From (2.3), $P_1 = O(R^{-4})$ and from the solution at the next order we can show that $P_2 = O(R^{-2})$. When the asymptotic scheme was extended to third order, it was found that P_3 was asymptotically proportional to $\log R$. At this order the pressure no longer decays as one leaves the gap, but must match with some outer solution. We therefore feel that extending the present scheme to third order will require a full solution both inside and outside the gap and that the derivation of the singular behaviour of the force and stresslet to third order cannot be justified on the basis of the gap solution alone. The earlier calculation of Jeffrey (1982) of the force to third order did not encounter this difficulty because the stream function does not show the same behaviour. Presumably the third order results found there are correct, but additional justification would be welcome.

The forces exerted by the spheres must, by symmetry, be in the z direction so we can calculate them from:

$$F_z = -\int \mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}S. \tag{2.4}$$

The known expressions for $X^A_{\alpha\beta}$ are reproduced. A new result is obtain by calculating the stresslet. We know from symmetry that the stresslets will have the form (Batchelor and Green, 1972; Kim and Mifflin, 1985):

$$S_1 = 4\pi a^2 X^G_{11}(\mathbf{kk}-\tfrac{1}{3}\mathbf{I})U \tag{2.5}$$

and

$$S_2 = \pi(a+b)^2 X^G_{21}(\mathbf{kk}-\tfrac{1}{3}\mathbf{I})U. \tag{2.6}$$

Therefore, we can obtain X^G_{11} by calculating the $\mathbf{kk}$ component of the stresslet. The integral is:

$$\tfrac{8}{3}\pi a^2 U X^G_{11} = -\int (\mathbf{k}\cdot\mathbf{n}\,\mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} - \tfrac{1}{3}\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n})\,\mathrm{d}S, \tag{2.7}$$

provided that the stress $\boldsymbol{\sigma}$ is expressed with respect to the non-dimensionalized coordinates. The integrand in this integral has the same behaviour for large R as the integrands previously considered (O'Neill and Stewartson, 1967; Jeffrey and Onishi, 1984a) and therefore the singular behaviour of X^G_{11} can be extracted directly from the asymptotic behaviour of the integrand in (2.7) following exactly the steps of the earlier papers. The result is best expressed using $\lambda = b/a = -1/\alpha$:

$$X^G_{11} = \frac{3\lambda^2}{2(1+\lambda)^2}\varepsilon^{-1} + \frac{3\lambda(1+12\lambda-4\lambda^2)}{10(1+\lambda)^3}\ln\varepsilon^{-1} + O(1). \tag{2.8}$$

In the same way, we obtain from the stresslet of the stationary sphere:

$$X^G_{21} = \frac{6\lambda^3}{(1+\lambda)^4}\varepsilon^{-1} + \frac{6\lambda^2(\lambda^2+12\lambda-4)}{5(1+\lambda)^5}\ln \varepsilon^{-1} + O(1). \tag{2.9}$$

A point of interest, in passing, concerns the way these results were obtained using computer algebra. In earlier papers, the expansions (1.3)–(1.5) were substituted into (2.7) and the integral was separated into powers of ε. The asymptotic forms of the integrands were obtained after this separation. The present calculation proceeded by substituting the solutions directly into the integrand without preliminary separation, then the asymptotic expansion was obtained and only lastly were the powers of ε separated.

We have two checks that the calculations have been performed correctly. First, Jeffrey (1988) has estimated, by purely numerical means, the singular terms in the functions for the special case $\lambda = 1$. The expressions (2.8)–(2.9) agree with the numerical estimates. Secondly, the reciprocal theorem can be used to derive a relation between the newly obtained functions and the functions $X^A_{\alpha\beta}$ already known. We suppose that the two spheres are held stationary in an ambient rate of strain:

$$\mathscr{E} = E(\mathbf{kk} - \tfrac{1}{3}\mathbf{I}).$$

The force on sphere a will be in the z direction and equal to:

$$F = 4\pi a\mu X^A_{11}(-E)(a+h) - 2\pi(a+b)\mu X^A_{12}(-E)b + 4\pi a^2\mu[X^G_{11} + \tfrac{1}{4}(1+\lambda)^2 X^G_{21}](2E/3).$$

Since singular forces arise from relative motion of nearly touching surfaces, and since in this case the surfaces are stationary, we conclude that F must be non-singular. In other words:

$$X^G_{11} + \tfrac{1}{4}(1+\lambda)^2 X^G_{21} - \tfrac{3}{2}(1+\varepsilon)X^A_{11} + \tfrac{3}{4}\lambda(1+\lambda)X^A_{12} = O(1). \tag{2.10}$$

This identity can be confirmed from the results above. To complete these results we must find expressions for X^G_{12} and X^G_{22}. To obtain these, we re-express X^G_{11} and X^G_{21} as functions of $\xi = 2\varepsilon/(1+\lambda)$. Thus:

$$X^G_{11} = \frac{3\lambda^2}{(1+\lambda)^3}\xi^{-1} + \frac{3\lambda(1+12\lambda-4\lambda^2)}{10(1+\lambda)^3}\ln \xi^{-1} + O(1), \tag{2.11}$$

$$X^G_{21} = \frac{12\lambda^3}{(1+\lambda)^5}\xi^{-1} + \frac{6\lambda^2(\lambda^2+12\lambda-4)}{5(1+\lambda)^5}\ln \xi^{-1} + O(1). \tag{2.12}$$

For these functions we have results similar to those derived in Jeffrey and Onishi (1984b), namely:

$$X^G_{11}(\xi,\lambda) = -X^G_{22}(\xi,\lambda^{-1}) \quad \text{and} \quad X^G_{12}(\xi,\lambda) = -X^G_{21}(\xi,\lambda^{-1}). \tag{2.13}$$

Therefore:

$$X^G_{22} = -\frac{3\lambda}{(1+\lambda)^3}\xi^{-1} - \frac{3(\lambda^2+12\lambda-4)}{10(1+\lambda)^3}\ln \xi^{-1} + O(1). \tag{2.14}$$

$$X^G_{12} = -\frac{12\lambda^2}{(1+\lambda)^5}\xi^{-1} - \frac{6\lambda(1+12\lambda-4\lambda^2)}{5(1+\lambda)^5}\ln \xi^{-1} + O(1). \tag{2.15}$$

From these results we see that:

$$X^G_{11} + \tfrac{1}{4}(1+\lambda)^2 X^G_{12} = O(1). \tag{2.16}$$

This reflects the fact that when the two spheres move with identical velocities the stresslets of the spheres are not singular in the gap width.

3. DEFORMING SPHERES

We now consider the case in which sphere a deforms while its centre remains at rest and sphere b does not deform and is at rest. The velocity on the surface of sphere a is:

$$\mathbf{u} = -2aE(1+\varepsilon-z)\mathbf{k} - aE\mathbf{r}. \tag{3.1}$$

This gives a uniform rate of strain $E_{ij} = 2E\delta_{3i}\delta_{3j} - E\delta_{1i}\delta_{1j} - E\delta_{2i}\delta_{2j}$ throughout the sphere. The velocity scale for the problem is $\mathscr{V} = 2aE$. Translating the boundary conditions into stretched coordinates, we obtain:

$$W = -1-\varepsilon(1-z) \quad \text{and} \quad U = -\tfrac{1}{2}\varepsilon^{1/2}R.$$

Therefore the boundary conditions on the deforming sphere become:

$$W_1(R, H_1) = -1 \quad \text{and} \quad W_2(R, H_1) = \tfrac{1}{2}R^2 - \tfrac{1}{8}R^4[\partial W_1/\partial Z]_{Z=H_1}$$

together with:

$$U_1(R, H_1) = 0 \quad \text{and} \quad U_2(R, H_1) = -\tfrac{1}{2}R - \tfrac{1}{8}R^4[\partial U_1/\partial Z]_{Z=H_1}.$$

The boundary conditions on the non-deforming sphere are the same as in the previous section. The method of solution is also the same and again the pressure when R is large goes as $O(R^{-4}) + \varepsilon O(R^{-2})$. Since this flow problem has not been solved before, we have both the force and the stresslet to calculate. However, the force calculation does not give any new results, because the reciprocal theorem shows that the forces are proportional to $X^G_{\alpha\beta}$ (the resistance matrix is symmetric). It does, of course, provide a second calculation of the functions obtained in the previous section and therefore provides a check on the working.

From the definition of $X^M_{\alpha\beta}$ the **kk** component of the stresslet of sphere a is:

$$S_{zz} = \frac{20}{3}\pi a^3 \mu X^M_{11} E,$$

and after completing the calculation, we find:

$$X^M_{11} = \frac{3\lambda^2}{5(1+\lambda)^2}\varepsilon^{-1} + \frac{3\lambda(1+17\lambda-9\lambda^2)}{25(1+\lambda)^3}\ln \varepsilon^{-1} + O(1). \tag{3.2}$$

The stresslet of sphere b is given by:

$$S_{zz} = \tfrac{5}{6}\pi(a+b)^3 \mu X^M_{21} E,$$

and the calculation shows that:

$$X^M_{21} = \frac{24\lambda^3}{5(1+\lambda)^5}\varepsilon^{-1} + \frac{24\lambda^2(-4\lambda^2+17\lambda-4)}{25(1+\lambda)^6}\ln \varepsilon^{-1} + O(1). \tag{3.3}$$

In the special case $\lambda = 1$, we recover the results obtained in Jeffrey (1988). If we again change to $\xi = 2\varepsilon/(1+\lambda)$, we obtain:

$$X^M_{22}(\xi, \lambda) = X^M_{11}(\xi, \lambda^{-1}) = \frac{6\lambda}{5(1+\lambda)^3}\xi^{-1} + \frac{3(\lambda^2+17\lambda-9)}{25(1+\lambda)^3}\ln \xi^{-1} + O(1), \tag{3.4}$$

$$X^M_{12}(\xi, \lambda) = X^M_{21}(\xi, \lambda) = \frac{48\lambda^3}{5(1+\lambda)^6}\xi^{-1} + \frac{24\lambda^2(-4\lambda^2+17\lambda-4)}{25(1+\lambda)^6}\ln \xi^{-1} + O(1), \tag{3.5}$$

The last result is consistent with the reciprocal theorem. For the special case $\lambda = 1$, the singular terms in X^M_{11} are the same as those in X^M_{12}, but we can see that in general this is not true. Since we expect non-singular behaviour when the spheres are at rest in an ambient rate of strain field, we obtain the relation:

$$X^M_{11} + \tfrac{1}{8}(1+\lambda)^3 X^M_{21} = \tfrac{2}{5}(1+\lambda)X^G_{11} + \mathrm{O}(1), \tag{3.6}$$

which can be verified from the results above.

4. NUMERICAL RESULTS

Numerical data for $X^G_{\alpha\beta}$ and $X^M_{\alpha\beta}$ have been published for equal spheres by Kim and Mifflin (1985) and by Jeffrey (1988). We can use them to check our results. After the series in Jeffrey (1988) have been improved as detailed below, the data of Kim and Mifflin (1985) are accurately reproduced and in addition, we obtain numerical values for the constants O(1) appearing in (2.8)–(2.9) and (3.2)–(3.3); these constants cannot be calculated from the asymptotic analysis. The present results can also be used to investigate the convergence of the series in Jeffrey (1988).

Each resistance function has been obtained in Jeffrey (1988) as a series in the variable t, the inverse separation of the sphere centres, given in terms of ε by the relation:

$$t = 1/(2+\varepsilon). \tag{4.1}$$

The series for X^G_{11} can be written:

$$X^G_{11} = \sum_{\substack{p\,\mathrm{odd}\\ p=1}}^{\infty} A_p t^p, \tag{4.2}$$

where the A_p are coefficients known up to A_{350}. Noting (2.8) and the identities given in Corless and Jeffrey (1988), we can rewrite the series as:

$$X^G_{11} = 2g_1 t/(1-4t^2) + g_2 \ln \frac{1+2t}{1-2t} + \sum_{\substack{p\,\mathrm{odd}\\ p=1}}^{\infty} (2^{-p}A_p - g_1 - 2g_2/p)(2t)^p, \tag{4.3}$$

where $g_1 = 3/8$ and $g_2 = 27/80$. In the same way we obtain:

$$X^G_{12} = -g_1/(1-4t^2) + g_1 + g_2 \ln (1-4t^2) - \sum_{\substack{p=2\\ p\,\mathrm{even}}}^{\infty} (2^{-p}A_p - g_1 - 2g_2/p)(2t)^p. \tag{4.4}$$

Similar expresssions can be written down for X^M_{11} and X^M_{12}. In fact,

$$X^M_{11} = g_3/(1-4t^2) - g_3 - g_4 \ln (1-4t^2) + 1 + \sum_{\substack{p=2\\ p\,\mathrm{even}}}^{\infty} (2^{-p}B_p - g_3 - 2g_4/p)(2t)^p, \tag{4.5}$$

and

$$X^M_{12} = 2g_3 t/(1-4t^2) + g_4 \ln \frac{1+2t}{1-2t} + \sum_{\substack{p\,\mathrm{odd}\\ p=1}}^{\infty} (2^{-p}B_p - g_3 - 2g_4/p)(2t)^p, \tag{4.6}$$

where $g_3 = 3/20$ and $g_4 = 27/200$. When the series above are summed to 350 terms, the numerical data are in good agreement with Kim and Mifflin (1985) but this in itself is not a very sensitive test of the convergence and accuracy of the new series expressions. It has been pointed out in Jeffrey and Onishi (1984b) that a more sensitive test is obtained by using the series to calculate the constants that are O(1) in the asymptotic forms.

Before we proceed to calculate the O(1) constants, we wish to extend the results (2.11)–(2.16) and (3.2)–(3.6) in the special case $\lambda = 1$ using a plausible argument. We know that the functions $X^A_{\alpha\beta}$ contain singular terms in ε^{-1}, $\ln \varepsilon$ and $\varepsilon \ln \varepsilon$ and we expect that the new functions studied here will be the same. We also note from (2.11)–(2.14) that the coefficients of the ε^{-1} and $\ln \varepsilon$ terms of X^G_{11} and X^G_{12} are equal. If we suppose that this will be true for the $\varepsilon \ln \varepsilon$ terms also, we can remove X^G_{12} from (2.10) and the equation then determines X^G_{11}. Thus,

$$X^G_{11}(\varepsilon, 1) = \tfrac{3}{2}(1+\tfrac{1}{2}\varepsilon)X^A_{11}. \tag{4.7}$$

Since Jeffrey (1982) obtained the $\varepsilon \ln \varepsilon$ term for X^A_{11}, we see that we get:

$$X^G_{11} = \frac{3}{8}\varepsilon^{-1} + \frac{27}{80}\ln \varepsilon^{-1} + G^X_{11}(1) + \frac{117}{560}\varepsilon \ln \varepsilon^{-1} + K_1\varepsilon. \tag{4.8}$$

The constants G^X_{11} and K_1 will be obtained from (4.3). In the same way, we get:

$$X^G_{12} = -\frac{3}{8}\frac{1}{\varepsilon} - \frac{27}{80}\ln \varepsilon^{-1} + G^X_{12}(1) - \frac{117}{560}\varepsilon \ln \varepsilon^{-1} + K_2\varepsilon. \tag{4.9}$$

The same considerations and assumptions lead to:

$$\begin{aligned} X^M_{11} &= \frac{2}{5}\left(1+\frac{1}{2}\varepsilon\right)X^G_{11} \\ &= \frac{3}{20}\varepsilon^{-1} + \frac{27}{200}\ln \varepsilon^{-1} + M^X_{11}(1) + \frac{423}{2800}\varepsilon \ln \varepsilon^{-1} + K_4\varepsilon, \end{aligned} \tag{4.10}$$

and a similar expression for X^M_{12}. It should be noted that this method cannot be used when the spheres are not equal, because (3.2)–(3.5) show explicitly that the 11 and 12 functions are not equal when $\lambda \neq 1$.

To obtain series expressions for the O(1) constants defined above, we substitute (4.1) into (4.3)–(4.6) and expand the singular terms for small ε. Removing the ε^{-1} and log ε terms and then setting $\varepsilon = 0$ gives us:

$$G^X_{11} = \boldsymbol{g}_1/4 + \boldsymbol{g}_2 \ln 4 - 2\boldsymbol{g}_5 + \sum_{\substack{p=1 \\ p\,\text{odd}}}^{\infty} \left[2^{-p}A_p - \boldsymbol{g}_1 - 2\boldsymbol{g}_2/p + \frac{4\boldsymbol{g}_5}{p(p+2)}\right], \tag{4.11}$$

$$G^X_{12} = \boldsymbol{g}_1/4 + \boldsymbol{g}_3 - \sum_{\substack{p=2 \\ p\,\text{even}}}^{\infty} \left[2^{-p}A_p - \boldsymbol{g}_1 - 2\boldsymbol{g}_2/p + \frac{4\boldsymbol{g}_5}{p(p+2)}\right], \tag{4.12}$$

$$M^X_{11} = -\boldsymbol{g}_3/4 - \boldsymbol{g}_6 + 1 + \sum_{\substack{p=2 \\ p\,\text{even}}}^{\infty} \left[2^{-p}B_p - \boldsymbol{g}_3 - 2\boldsymbol{g}_4/p + \frac{4\boldsymbol{g}_6}{p(p+2)}\right], \tag{4.13}$$

$$M^X_{12} = \boldsymbol{g}_3/4 + \boldsymbol{g}_4 \ln 4 - 2\boldsymbol{g}_6 + \sum_{\substack{p=1 \\ p\,\text{odd}}}^{\infty} \left[2^{-p}B_p - \boldsymbol{g}_3 - 2\boldsymbol{g}_4/p + \frac{4\boldsymbol{g}_6}{p(p+2)}\right]. \tag{4.14}$$

Here, $\boldsymbol{g}_5 = 117/560$ and $\boldsymbol{g}_6 = 423/2800$.

In Table 1, these constants are shown evaluated by summing the series to various numbers of terms. In addition the constants $\boldsymbol{g}_5$ and $\boldsymbol{g}_6$ were first set to zero and then given their proper values. It can be seen that these terms usefully improve the convergence of the series and therefore should be found for all λ, although their calculation will require a fair amount

Table 1. Convergence properties of series for $G^X_{\alpha\beta}$ and $M^X_{\alpha\beta}$

	G^X_{11}		G^X_{12}	
n	without $\boldsymbol{g}_5$	with $\boldsymbol{g}_5$	without $\boldsymbol{g}_5$	with $\boldsymbol{g}_5$
100	−0.46507	−0.46921	0.19141	0.19551
200	−0.46723	−0.46931	0.19331	0.19538
300	−0.46789	−0.46928	0.19403	0.19541
350	−0.46808	−0.46927	0.19424	0.19542
	M^X_{11}		M^X_{12}	
n	without $\boldsymbol{g}_6$	with $\boldsymbol{g}_6$	without $\boldsymbol{g}_6$	with $\boldsymbol{g}_6$
100	0.71927	0.71631	−0.14194	−0.14493
200	0.71858	0.71708	−0.14424	−0.14574
300	0.71815	0.71715	−0.14480	−0.14581
350	0.71805	0.71719	−0.14498	−0.14585

of work. Finally, by fitting the numerical data of Kim and Mifflin (1985) to (4.8)–(4.10) we can obtain values for the O(ε) terms as well:

$$X^G_{11} = \frac{3}{8}\varepsilon^{-1} + \frac{27}{80}\ln\varepsilon^{-1} - 0.4693 + \frac{117}{560}\varepsilon\ln\varepsilon^{-1} + 0.36\varepsilon,$$

$$X^G_{12} = -\frac{3}{8}\varepsilon^{-1} - \frac{27}{80}\ln\varepsilon^{-1} + 0.1954 - \frac{117}{560}\varepsilon\ln\varepsilon^{-1} - 0.29\varepsilon,$$

$$X^M_{11} = \frac{3}{20}\varepsilon^{-1} + \frac{27}{200}\ln\varepsilon^{-1} + 0.7172 + \frac{423}{2800}\varepsilon\ln\varepsilon^{-1} + 0.07\varepsilon,$$

$$X^M_{12} = \frac{3}{20}\varepsilon^{-1} + \frac{27}{200}\ln\varepsilon^{-1} - 0.1459 + \frac{423}{2800}\varepsilon\ln\varepsilon^{-1} + 0.15\varepsilon.$$

Acknowledgement—This work was supported by the Natural Science and Engineering Research Council of Canada.

REFERENCES

G. K. Batchelor (1970) The stress system in a suspension of force-free particles. *J. Fluid Mech.* **41,** 545.

G. K. Batchelor and J. T. Green (1972) The hydrodynamic interaction of two small freely-moving spheres in a linear flow field. *J. Fluid Mech.* **56,** 375.

H. Bremner and M. E. O'Neill (1972) On the Stokes resistance of multiparticle systems in a linear shear field. *Chem. Engng Sci.* **27,** 1421.

M. D. A. Cooley and M. E. O'Neill (1969) On the slow motion generated in a viscous fluid by the approach of a sphere to a plane wall or a stationary sphere. *Mathematica* **16,** 37.

R. M. Corless and D. J. Jeffrey (1988) Stress moments of nearly touching spheres in low Reynolds number flow. *Z. angew. Math. Phys.* (in press).

R. E. Hansford (1970) On converging solid spheres in a highly viscous fluid. *Mathematica* **17,** 250.

D. J. Jeffrey (1982) Low Reynolds-number flow between converging spheres. *Mathematica* **29,** 58.

D. J. Jeffrey (1988) Interactions between two equal spheres at low Reynolds numbers. *Archives Mech.* (submitted).

D. J. Jeffrey and Y. Onishi (1984a) The forces and couples acting on two nearly touching spheres in low-Reynolds-number flow. *Z. angew. Math. Phys.* **35,** 634.

D. J. Jeffrey and Y. Onishi (1984b) Calculation of the resistance and mobility functions for two unequal spheres in low-Reynolds-number flow. *J. Fluid Mech.* **139,** 261.

S. Kim and R. T. Mifflin (1985) The resistance and mobility functions of two equal spheres in low-Reynolds-number flow. *Phys. Fluid* **28,** 2033.

M. E. O'Neill and K. Stewartson (1967) On the slow motion of a sphere parallel to a nearby plane wall. *J. Fluid Mech.* **27,** 705.

R. Schmitz and B. U. Felderhof (1982) Friction matrix for two spherical particles with hydrodynamic interaction. *Physica* **113A,** 103.