The Lambert W Function

Robert M. Corless and David J. Jeffrey

1 Definition and Basic Properties

For a given complex number z, the equation

$$we^w = z$$

has a countably infinite number of solutions, which are denoted by $W_k(z)$ for integers k. Each choice of k specifies a *branch* of the Lambert W function. By convention, only the branches $k = 0$ (called the *principal* branch) and $k = -1$ are real-valued for any z; the range of every other branch excludes the real axis, although the range of $W_1(z)$ includes $(-\infty, -1/e]$ in its closure. Only $W_0(z)$ contains positive values in its range (see figure 1). When $z = -1/e$ (the only nonzero branch point), there is a double root $w = -1$ of the basic equation $we^w = z$. The conventional choice of branches assigns

$$W_0(-1/e) = W_{-1}(-1/e) = -1$$

and implies that $W_1(-1/e - i\varepsilon^2) = -1 + O(\varepsilon)$ is arbitrarily close to -1, because the conventional branch choice means that the point -1 is on the border between these three branches. Each branch is a single-valued complex function, analytic away from the branch point and branch cuts.

The set of all branches is often referred to, loosely, as the Lambert W "function"; but of course W is multivalued. Depending on context, the symbol $W(z)$ can refer to the principal branch ($k = 0$) or to some unspecified branch. Numerical computation of any branch of W is typically carried out by Newton's method or a variant thereof. Images of $W_k(re^{i\theta})$ for various k, r, and θ are shown in figure 2.

In contrast to more commonly encountered multi-branched functions, such as the inverse sine or cosine, the branches of W are not linearly related. However, by rephrasing things slightly, in terms of the *unwinding number*

$$\mathcal{K}(z) := \frac{z - \ln(e^z)}{2\pi i}$$

and the related *single-valued* function

$$\omega(z) := W_{\mathcal{K}(z)}(e^z),$$

which is called the *Wright ω function*, we do have the somewhat simple relationship between branches that $W_k(z) = \omega(\ln_k z)$, where $\ln_k z$ denotes $\ln z + 2\pi ik$ and $\ln z$ is the principal branch of the logarithm, having $-\pi < \mathrm{Im}(\ln z) \leqslant \pi$.

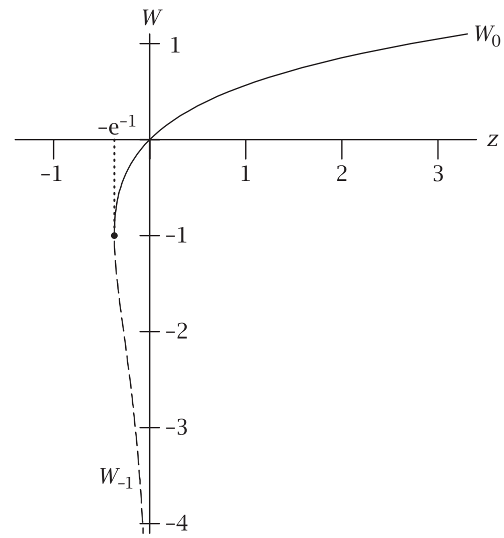

Figure 1 Real branches of the Lambert W function. The solid line is the principal branch W_0; the dashed line is W_{-1}, which is the only other branch that takes real values. The small filled circle at the branch point corresponds to the one in figure 2.

The Wright ω function helps to solve the equation $y + \ln y = z$. We have that if $z \neq t \pm i\pi$ for $t < -1$, then $y = \omega(z)$. If $z = t - i\pi$ for $t < -1$, then there is no solution to the equation; if $z = t + i\pi$ for $t < -1$, then there are two solutions: $\omega(z)$ and $\omega(z - 2\pi i)$.

1.1 Derivatives

Implicit differentiation yields

$$W'(z) = e^{-W(z)}/(1 + W(z))$$

as long as $W(z) \neq -1$. The derivative can be simplified to the *rational* differential equation

$$\frac{dW}{dz} = \frac{W}{z(1 + W)}$$

if, in addition, $z \neq 0$. Higher derivatives follow naturally.

1.2 Integrals

Integrals containing $W(x)$ can often be performed analytically by the change of variable $w = W(x)$, used in an inverse fashion: $x = we^w$. Thus,

$$\int \sin W(x)\,dx = \int (1 + w)e^w \sin w\,dw,$$

and integration using usual methods gives

$$\tfrac{1}{2}(1 + w)e^w \sin w - \tfrac{1}{2}we^w \cos w,$$

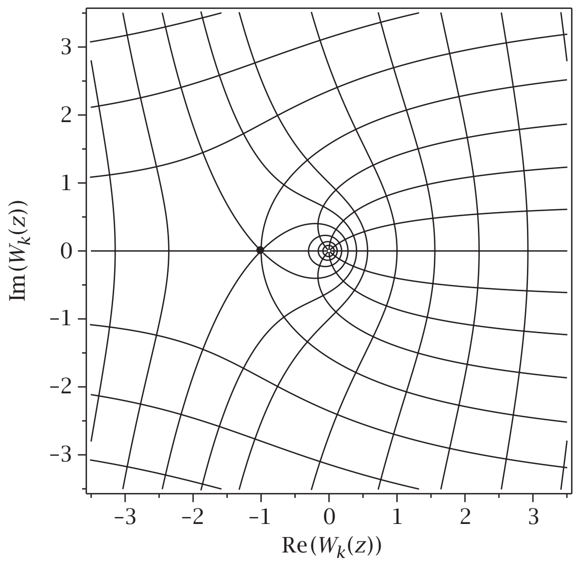

Figure 2 Images of circles and rays in the z-plane under the maps $z \to W_k(z)$. The circle with radius e^{-1} maps to a curve that goes through the branch point, as does the ray along the negative real axis. This graph was produced in Maple by numerical evaluation of $\omega(x+iy) = W_{\mathcal{K}(iy)}(e^{x+iy})$ first for a selection of fixed x and varying y, and then for a selection of fixed y and varying x. These two sets produce orthogonal curves as images of horizontal and vertical lines in x and y under ω, or, equivalently, images of circles with constant $r = e^x$ and rays with constant $\theta = y$ under W.

which eventually gives

$$\int 2\sin W(x)\,dx = \left(x + \frac{x}{W(x)}\right)\sin W(x) - x\cos W(x) + C.$$

More interestingly, there are many *definite* integrals for $W(z)$, including one for the principal branch that is due to Poisson and is listed in the famous table of integrals by D. Bierens de Haan. The following integral, which is of relatively recent construction and which is valid for z not in $(-\infty, -1/e]$, can be computed with spectral accuracy by the trapezoidal rule:

$$\frac{W(z)}{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 - v\cot v)^2 + v^2}{z + v\csc v\, e^{-v\cot v}}\,dv.$$

1.3 Series and Generating Functions

Euler was the first to notice, using a series due to Lambert, that what we now call the Lambert W function has a convergent series expansion around $z = 0$:

$$W(z) = \sum_{n\geqslant 1} \frac{(-n)^{n-1}}{n!} z^n.$$

Euler knew that this series converges for $-1/e < z \leqslant 1/e$. The nearest singularity is the branch point $z = -1/e$.

W can also be expanded in series about the branch point. The series at the branch point can be expressed most cleanly using the *tree function* $T(z) = -W(-z)$, rather than W or ω, but keeping with W we have

$$W_0(-e^{-1-z^2/2}) = -\sum_{n\geqslant 0} (-1)^n a_n z^n,$$
$$W_{-1}(-e^{-1-z^2/2}) = -\sum_{n\geqslant 0} a_n z^n,$$

where the a_n are given by $a_0 = a_1 = 1$ and

$$a_n = \frac{1}{(n+1)a_1}\left(a_{n-1} - \sum_{k=2}^{n-1} k a_k a_{n+1-k}\right).$$

These give an interesting variation on Stirling's formula for the asymptotics of $n!$. Euler's integral

$$n! = \int_0^\infty t^n e^{-t}\,dt$$

is split at the maximum of the integrand ($t = n$) and each integral is transformed using the substitutions $t = -nW_k(-e^{-1-z^2/2})$, where $k = 0$ is used for $t \leqslant n$ and $k = -1$ otherwise. The integrands then simplify to $t^n e^{-t} = n^n e^{-n} e^{-nz^2/2}$ and the differentials dt are obtained as series from the above expansions. Term-by-term integration leads to

$$n! \sim \frac{n^{n+1}}{e^n} \sum_{k\geqslant 0} (2k+1)a_{2k+1}\left(\frac{2}{n}\right)^{k+1/2} \Gamma(k + \tfrac{1}{2}),$$

where Γ is the gamma function.

Asymptotic series for $z \to \infty$ have been known since de Bruijn's work in the 1960s. He also proved that the asymptotic series are actually convergent for large enough z. The series begin as follows: $W_k(z) \sim \ln_k(z) - \ln(\ln_k(z)) + o(\ln\ln_k z)$. Somewhat surprisingly, these series can be reversed to give a simple (though apparently useless) expansion for the logarithm in terms of compositions of W:

$$\ln z = W(z) + W(W(z)) + W(W(W(z))) + \cdots + W^{(N)}(z) + \ln W^{(N)}(z)$$

for a suitably restricted domain in z. The series obtained by omitting the term $\ln W^{(N)}(z)$ is not convergent as $N \to \infty$, but for fixed N if we let $z \to \infty$ the approximation improves, although only tediously slowly.

2 Applications

Because W is a so-called implicitly elementary function, meaning it is defined as an implicit solution of an equation containing only elementary functions, it can be considered an "answer" rather than a question. That it solves a simple rational differential equation means that it occurs in a wide range of mathematical models. Out of many applications, we mention just two favorites.

First, a serious application. W occurs in a chemical kinetics model of how the human eye adapts to darkness after exposure to bright light: a phenomenon known as bleaching. The model differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} O_p(t) = \frac{K_m O_p(t)}{\tau(K_m + O_p(t))},$$

and its solution in terms of W is

$$O_p(t) = K_m W\left(\frac{B}{K_m} \mathrm{e}^{B/K_m - t/\tau}\right),$$

where the constant B is the initial value of $O_p(0)$: that is, the amount of initial bleaching. The constants K_m and τ are determined by experiment. The solution in terms of W enables more convenient analysis by allowing the use of known properties.

The second application we mention is nearly frivolous. W can be used to explore solutions of the so-called astrologer's equation, $\dot{y}(t) = ay(t+1)$. In this equation, the rate of change of y is supposed to be proportional to the value of y one time unit into the *future*. Dependence on past times instead leads to delay differential equations, which of course are of serious interest in applications, and again W is useful there in much the same way as for this frivolous problem.

Frivolity can be educational, however. Notice first that if $\mathrm{e}^{\lambda t}$ satisfies the equation, then $\lambda = a\mathrm{e}^{\lambda}$, and therefore $\lambda = -W_k(-a)$. For the astrologer's equation, any function $y(t)$ that can be expressed as a *finite* linear combination $y(t) = \sum_{k \in M} c_k \mathrm{e}^{-W_k(-a)t}$ for $0 \leqslant t \leqslant 1$ and some finite set M of integers then solves the astrologer's equation for all time. Thus, perfect knowledge of $y(t)$ on the time interval $0 \leqslant t \leqslant 1$ is sufficient to predict $y(t)$ for all time. However, if the knowledge of $y(t)$ is imperfect, even by an infinitesimal amount (omitting a single term $\varepsilon \mathrm{e}^{-W_K(-a)t}$, say, where K is some large integer), then since the real parts of $-W_K(-a)$ go to infinity as $K \to \infty$ by the first two terms of the logarithmic series for W_k given above, the "true" value of $y(t)$ can depart arbitrarily rapidly from the prediction. This seems completely in accord with our intuitions about horoscopes.

Returning to serious applications, we note that the tree function $T(z)$ has huge combinatorial significance for all kinds of enumeration. Many instances can be found in Knuth's selected papers, for example. Additionally, a key reference to the tree function is a note by Borel in *Comptes Rendus de l'Académie des Sciences* (volume 214, 1942; reprinted in his *Œuvres*). The generating function for probabilities of the time between periods when a queue is empty, given Poisson arrivals and service time σ, is $T(\sigma \mathrm{e}^{-\sigma} z)/\sigma$.

3 Solution of Equations

Several equations containing algebraic quantities together with logarithms or exponentials can be manipulated into either the form $y + \ln y = z$ or $w\mathrm{e}^w = z$, and hence solved in terms of the Lambert W function. However, it appears that not every exponential polynomial equation—or even most of them—can be solved in this way. We point out one equation, here, that starts with a nested exponential and can be solved in terms of branch *differences* of W: a solution of

$$z + v \csc v \mathrm{e}^{-v \cot v} = 0$$

is $v = (W_k(z) - W_\ell(z))/(2\mathrm{i})$ for some pair of integers k and ℓ; moreover, every such pair generates a solution. This bi-infinite family of solutions has accumulation points of zeros near odd multiples of π, which in turn implies that the denominator in the above definite integral for $W(z)/z$ has essential singularities at $v = \pm\pi$. This example underlines the importance of the fact that the branches of W are not trivially related.

Another equation of popular interest occurs in the analysis of the limit of the recurrence relation

$$a_{n+1} = z^{a_n}$$

starting with, say, $a_0 = 1$. This sequence has $a_1 = z$, $a_2 = z^z$, $a_3 = z^{z^z}$, and so on. If this limit converges, it does so to a solution of the equation $a = z^a$. By inspection, the limit that is of interest is $a = -W(-\ln z)/\ln z$. Somewhat surprisingly, this recurrence relation—which defines the so-called tower of exponentials—diverges for small enough z, even if z is real. Specifically, the recurrence converges for $\mathrm{e}^{-\mathrm{e}} \leqslant z \leqslant \mathrm{e}^{1/\mathrm{e}}$ if z is real and diverges if $z < \mathrm{e}^{-\mathrm{e}} = 0.0659880\ldots$. This fact was known to Euler. The detailed convergence properties for complex z were settled only relatively recently. Describing the regions in the complex plane where the recurrence relation converges to an n-cycle is made possible by a transformation that is itself related to W: if $\zeta = -W(-\ln z)$, then the iteration converges if $|\zeta| < 1$, and also if $\zeta = e^{i\theta}$ for θ equal to some *rational* multiple of π, say $m\pi/k$. Regions where the iteration converges to a k-cycle may touch the unit circle at those points.

4 Retrospective

The Lambert W function crept into the mathematics literature unobtrusively, and it now seems natural there. There is even a matrix version of it, although the solution of the matrix equation $Se^S = A$ is not always $W(A)$.

Hindsight can, as it so often does, identify the presence of W in writings by Euler, Poisson, and Wright and in many applications. Its implementation in Maple in the early 1980s was a key step in its eventual popularity.

Indeed, its recognition and naming supports Alfred North Whitehead's opinion that:

> By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems.

Further Reading

Corless, R. M., G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. 1996. On the Lambert W function. *Advances in Computational Mathematics* 5(1):329–59.

Knuth, D. E. 2000. *Selected Papers on the Analysis of Algorithms*. Palo Alto, CA: Stanford University Press.

Knuth, D. E. 2003. *Selected Papers on Discrete Mathematics*. Stanford, CA: CSLI Publications.

Lamb, T., and E. Pugh. 2004. Dark adaptation and the retinoid cycle of vision. *Progress in Retinal and Eye Research* 23(3):307–80.

Olver, F. W. J., D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds. 2010. *NIST Handbook of Mathematical Functions*. Cambridge: Cambridge University Press. (Electronic version available at http://dlmf.nist.gov.)