The Wright ω function

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Abstract. This paper defines the Wright ω function, and presents some of its properties. As well as being of intrinsic mathematical interest, the function has a specific interest in the context of symbolic computation and automatic reasoning with nonstandard functions. In particular, although Wright ω is a cognate of the Lambert W function, it presents a different model for handling the branches and multiple values that make the properties of W difficult to work with. By choosing a form for the function that has fewer discontinuities (and numerical difficulties), we make reasoning about expressions containing such functions easier. A final point of interest is that some of the techniques used to establish the mathematical properties can themselves potentially be automated, as was discussed in a paper presented at AISC Madrid [3].

1 Notation and Definitions

The Wright ω function is a single-valued function, defined in terms of the Lambert W function. Lambert W satisfies $W(z)\exp(W(z)) = z$, and has an infinite number of branches, denoted $W_k(z)$, for $k \in \mathbb{Z}$. See [4] for a discussion of why the branches were chosen as they are. The Lambert W function is therefore multivalued. The Wright ω function[1] is a single-valued function, defined as follows:

$$\omega(z) = W_{\mathcal{K}(z)}\left(e^z\right) \tag{1}$$

where $\mathcal{K}(z) = \lceil(\mathrm{Im}(z) - \pi)/(2\pi)\rceil$ is the *unwinding number* of z. Note that the sign of this unwinding number is such that $\ln(\exp(z)) = z + 2\pi i\mathcal{K}(z)$, which is opposite to the sign used in [5], because we discovered after that publication that the present sign choice leads to fewer minus signs in formulas.

[1] This nomenclature has never, to our knowledge, appeared in print before. We use the letter ω as a cognate of W, and we name this function after Sir Edward M. Wright, for his works [12] establishing the complex branching behaviour of this function as a tool for investigating the roots of $y\exp(y) = z$ (later called the Lambert W function)

2 Graphs and special values

A graph of $\omega(z)$ for real z can be produced easily in Maple by the command `plot([y+ln(y),y,y=0.001..2]);`. A section of the Riemann surface for $\omega(z)$ can be plotted by the following commands:

```
omega := mu + I*nu;
x := evalc(Re(omega+ln(omega)));
y := evalc(Im(omega+ln(omega)));
plot3d( [x,y,mu], mu=-4..2, nu=-4..4,
        colour=black, axes=BOXED,
        style=PATCHNOGRID, labels=["x","y","mu"],
        view=[-2..1, -5..5, -5..3],
        grid=[200,200], style=POINT );
```

See Table 1 for special values.

Table 1. Special values of $\omega(z)$.

z	$\omega(z)$
$-\infty$	0
0	$W_0(1)$
1	1
$2+\ln 2$	2
$-1/3+\ln(1/3)+i\pi$	$-1/3 = W_0\left(-\frac{1}{3}e^{-1/3}\right)$
$-1/3+\ln(1/3)-i\pi$	$W_{-1}\left(-\frac{1}{3}e^{-1/3}\right)$
$-1+i\pi$	-1
$-1-i\pi$	-1
$-2+\ln 2+i\pi$	$W_0\left(-2e^{-2}\right)$
$-2+\ln 2-i\pi$	$-2 = W_{-1}\left(-2e^{-2}\right)$
∞	∞

2.1 Summary of Results of This Note.

The main result is a clarification, using this new function, of results due originally to Wright [12] and independently rediscovered in [11] and [9]. Although $y=\omega(z)$ satisfies the equation (in this paper $\ln(z)$ is the principal branch of the logarithm of z)

$$y+\ln y = z\,, \tag{2}$$

when $z\neq t\pm i\pi$ for $t\leq -1$, there should be a distinction made between the solutions of the equation, and Wright ω. In other words, (2) is not a satisfactory definition of ω.

In addition to this basic point, we here present new branch point series (with the correct closure), new asymptotic series (from the equivalent series for the Lambert W function), and new proofs of the analytic properties of $\omega(z)$, using properties of the unwinding number.

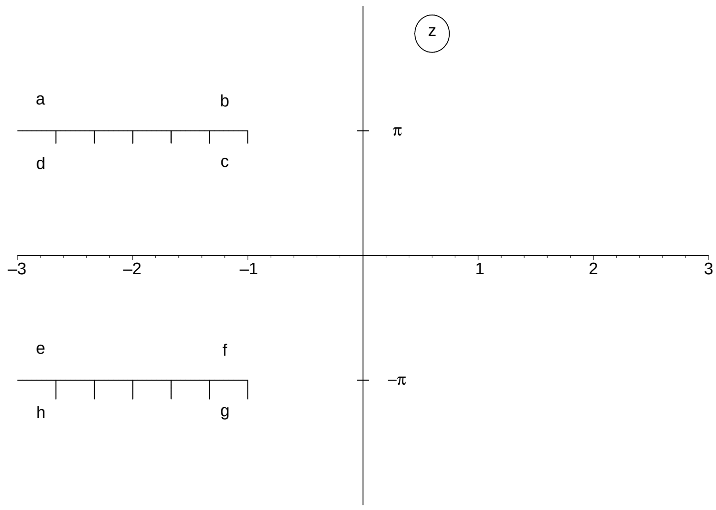

Fig. 1. The z-plane, showing the slit (equivalently, branch cut) we call the "doubling line" (above) and its "reflection", across each of which the Wright ω function is discontinuous. Along both slits, the closure (indicated by short lines extending down from the slits) is taken from below—clockwise around the branch points—to agree with the closure of the unwinding number.

We here summarize some properties of ω, proved in [9]. First, equation (2) has a unique solution, $\omega(z)$, for all $z \in \mathbb{C}$ except on the line L_D defined by $z = t \pm i\pi$ for $t \leq -1$. When z is on L_D, the equation has precisely two solutions, these being $\omega(z)$ and $\omega(z - 2\pi i)$; we therefore call L_D the "doubling line". See Figure 1 and Figure 2. On the reflection of the doubling line, namely, the line defined by $z = t - i\pi$, with $t \leq -1$, equation (2) has no solution at all[2]. Second, ω is an analytic function of z except on the doubling line *and its reflection* $z = t - i\pi$ for $t \leq -1$, where $\omega(z)$ is discontinuous. This immediately gives the following.

[2] Unfortunately, in the paper [6], we got this wrong—we missed the fact that there was no solution on this line. Indeed, at that time, we hadn't realized this function is discontinuous there. Additionally, we were using the opposite sign for the unwinding number, which made the formulas messier.

Theorem: For all $z \in \mathbb{C}$ and integers k,

$$W_k(z) = \omega(\ln_k(z)), \tag{3}$$

where $\ln_k(z) = \ln z + 2\pi i k$. [This logarithmic notation is discussed further in a later section.]

Proof. This holds at least provided z is not in the interval $-\exp(-1) \leq z < 0$ and $k = -1$, which is the image in the domain of W of the critical doubling line (and also the image of its reflection). If z is in the interval $-\exp(-1) \leq z < 0$, and $k = -1$, then we have instead that $W_0(z) = \omega(\ln|z| + i\pi)$ since $\mathcal{K}(\ln|z| + i\pi) = 0$, and that $W_{-1}(z) = \omega(\ln|z| - i\pi)$ since $\mathcal{K}(\ln|z| - i\pi) = -1$. Phrasing this the other way, we have

$$W_0(z) = \omega(\ln z)$$

and

$$W_{-1}(z) = \omega(\ln z - 2\pi i)\ .$$

$\natural$

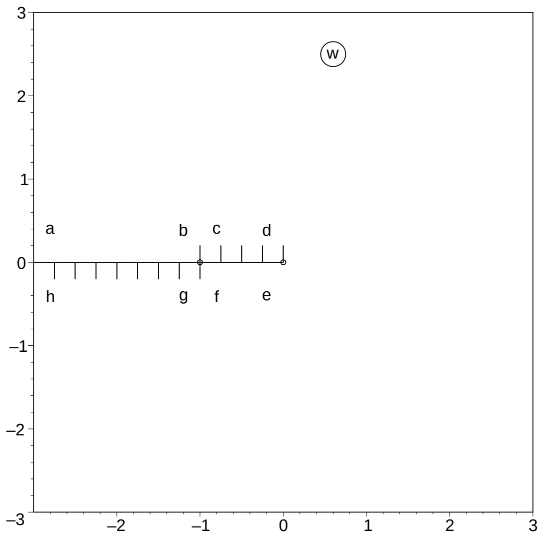

Fig. 2. The ω-plane, showing the images of doubling slit and its reflection. The negative real ω-axis is not, *per se*, a branch cut (this is the range of the function) but it is a branch cut of $\omega + \ln \omega$, which is why that expression is not exactly the inverse function for ω.

2.2 Properties of ω

We group the properties into analytic properties and algebraic properties.

Analytic properties Theorems and lemmas:

(i) $\omega(z)$ is single-valued

(ii) $\omega : \mathbb{C} \to \mathbb{C}$ is onto $\mathbb{C} \setminus \{0\}$.

(ii)(a) Except at $z = -1 \pm i\pi$, where $\omega(z) = -1$, $\omega : \mathbb{C} \to \mathbb{C}$ is injective; hence ω^{-1} exists uniquely except at 0 and -1.

(iii) See Figure 2.

$$\omega^{-1}(y) = \begin{cases} y + \ln(y) - 2\pi i & -\infty < y < -1 \\ -1 \pm i\pi & y = -1 \\ y + \ln(y) & \text{otherwise.} \end{cases}$$

(iv) (a) ω is continuous (in fact analytic) except at $z = t \pm i\pi$ for $t \leq -1$.

(b) For $z = t \pm i\pi$ and $t \leq -1$, we have

(1) $\quad \omega(t + i\pi^-) = \omega(t + i\pi) = \omega(t - i\pi^-)$

(2) $\quad \omega(t + i\pi^+) = \omega(t - i\pi) = \omega(t - i\pi^+)$

(v) (a) $\omega + \ln\omega = z \Longleftrightarrow \mathcal{K}(\omega + \ln\omega) = \mathcal{K}(z)$.

(b) $\mathcal{K}(\omega + \ln\omega) = \mathcal{K}(z)$ unless $z = t - i\pi$, $t \leq -1$.

(vi) If $z \neq t - i\pi$ for $t \leq -1$, then $\omega(z) + \ln\omega(z) = z$.

If moreover $z \neq t + i\pi$, $t \leq -1$, then this solution is unique; if $z = t + i\pi$, $t \leq -1$, then $y = \omega(t - i\pi)$ is also satisfies $y + \ln y = z$. There is no y such that $y + \ln y = z$ if $z = t - i\pi$.

2.3 Proofs.

(i) The functions $\exp z$, $\mathcal{K}(z)$ and $W_k(z)$ (for each fixed k) are single-valued. Hence the composition $W_{\mathcal{K}(z)}(\exp z)$ is single-valued.

(ii) $\mathcal{K}(z)$ covers all of $\mathbb{Z}$ as z covers all of $\mathbb{C}$, and the branches of W partition the plane, except that -1 is hit twice: $W_{-1}(-1/e) = W_0(-1/e) = -1$. Only $W_0(0) = 0$, and 0 is the only point not in the range of e^z; hence there is no finite z such that $\omega(z) = 0$, but no other points in the range are missed.

(iii) (v) $\Longrightarrow$ (iii) because $\omega e^\omega = e^z$, and hence $\ln(\omega e^\omega) = \ln\omega + \omega - 2\pi i\mathcal{K}(\omega + \ln\omega) = z - 2\pi i\mathcal{K}(z)$ and since $\mathcal{K}(\omega + \ln\omega) = \mathcal{K}(z)$ except on $z = t - i\pi$ for $t \leq -1$, we have $z = \omega + \ln\omega$ for the "otherwise" case; the case $y = -1$ is by computation; and the case where $z = t - i\omega$, $\omega(z) \in (-\infty, -1) \Longleftrightarrow \mathcal{K}(z) = -1$ and direct computation from $\omega e^\omega = e^z \in (-1/e, 0)$ gives $z = \omega + \ln(-\omega) - i\pi$ as claimed. Equivalently, $\overline{z} = \omega + \ln(\omega)$.

(iv) (iii) $\Longrightarrow$ (iv) because ω is the inverse of $y \to y + \ln(y)$ except when $-\infty < y \leq -1$ when ω has a different inverse. Moreover, $y + \ln y$ is continuous except when $y \leq 0$. Its derivative is $1 + 1/y$, which is zero only if $y = -1$. Therefore ω is continuous (analytic) except possibly when $\omega < 0$. This is precisely $z = t \pm i\pi$, $t \leq 1$. Inspection shows that ω really is discontinuous on $z = t \pm i\pi$ for $t < 1$; but $\omega(t + i\pi) = \omega(t + i\pi^-)$ and $\omega(t - i\pi) = \omega(t - i\pi^+)$ are both continuous from below, because $\mathcal{K}(z)$ is. The fact that $\omega(t + i\pi^-) = \omega(t - i\pi^-)$ follows from the analyticity of $W_0(z)$ in $|z| < 1/e$.

(v) (a) $W_{\mathcal{K}(z)}(e^z)e^{W_{\mathcal{K}(z)}(e^z)} = e^z = \omega(z)e^{\omega(z)}$ by definition. Taking logs, $\ln(\omega e^\omega) = \ln e^z$, or $\omega + \ln\omega - 2\pi i\mathcal{K}(\omega + \ln\omega) = z - 2\pi i\mathcal{K}(z)$. Therefore, $\omega + \ln\omega = z \Longleftrightarrow \mathcal{K}(\omega + \ln\omega) = \mathcal{K}(z)$.

(v) (b) $\mathcal{K}(W_{\mathcal{K}(z)}(e^z) + \ln W_{\mathcal{K}(z)}(e^z)) = \mathcal{K}(z)$. $\mathcal{K}(a)$ can change only when $a = t + (2k+1)\pi$ for $k \in \mathbb{Z}$, or when a is itself discontinuous. We distinguish two cases, therefore:

(1) $W_{\mathcal{K}(z)}(e^z) + \ln W_{\mathcal{K}(z)}(e^z)$ can be discontinuous at discontinuities of $\mathcal{K}(z)$, namely $z = t + (2k+1)\pi$ for $k \in \mathbb{Z}$, or when $W_{\mathcal{K}(z)}(e^z) < 0$. We ignore discontinuities of $\mathcal{K}(z)$ for the moment. $W_{\mathcal{K}(z)}(e^z) < 0$ only when (i) $\mathcal{K}(z) = 0$ and $e^z < 0 \Longleftrightarrow z = t + i\pi$, $t \le -1$, or (ii) $\mathcal{K}(z) = -1$ and $e^z < 0 \Longleftrightarrow z = t - i\pi$, $t \le -1$. Both (i) and (ii) are discontinuities of $\mathcal{K}(z)$ anyway.

(2) $\mathcal{K}(\omega(z) + \ln\omega(z))$ can be discontinuous when $\omega + \ln\omega = t + (2k+1)\pi i \Longrightarrow \omega e^\omega = -e^t \Longleftrightarrow \omega(z) \subseteq$ an image of $\mathbb{R}^-$ under W. Therefore $z \subseteq$ a pre-image of $\mathbb{R}^-$ under e^z.

But this is just $z = t + (2k+1)\pi i$, which is a place of discontinuity of $\mathcal{K}(z)$. Note that $\mathcal{K}$ is integral-valued. Therefore, if $\omega(z)$ is such that $\mathcal{K}(\omega(z) + \ln(\omega(z))) = \mathcal{K}(z)$ for any z in a strip $(2k-1)\pi < \mathrm{Im}z \le (2k+1)\pi$, where $\omega + \ln(\omega)$ is continuous, then we have $\mathcal{K}(\omega(z) + \ln\omega(z)) = \mathcal{K}(z)$ everywhere in that strip. Let us choose $k \in \mathbb{Z}$, and look at the pre-image of $\omega = 2k\pi i$. Then $\omega + \ln\omega = 2k\pi i + \ln(2k\pi) + i\pi/2$ and hence $\mathcal{K}(\omega + \ln\omega) = k$. Since $\omega = W_{\mathcal{K}(z)}(e^z)$ we have $\omega e^\omega = e^z$ and $2k\pi i \cdot e^{2k\pi i} = e^z \Longleftrightarrow e^z = 2k\pi i$; *moreover* $2k\pi i \in$ range $W_{\mathcal{K}(z)}$, and therefore $\mathcal{K}(z) = k$. Therefore

$$\begin{aligned} z &= \ln(2k\pi i) + 2k\pi i \\ &= \omega + \ln\omega. \end{aligned}$$

This establishes that if $\omega(z) = W_{\mathcal{K}(z)}(e^z)$, then $\omega + \ln\omega = z$ except possibly on the edges of the strips $z = t + (2k+1)\pi i$. Now we have $\mathcal{K}(\omega(z) + \ln(\omega(z))) = \mathcal{K}(z)$ if $(2k-1)\pi < \mathrm{Im}(z) < (2k+1)\pi$, and hence $\omega + \ln\omega = z$. Note that $\omega(z) = W_{\mathcal{K}(z)}(e^z)$ is continuous from below as $\mathrm{Im}(z) \to (2k+1)\pi^-$. Therefore, provided that $\omega(z) \notin \mathbb{R}^-$, $\omega(z) + \ln(\omega(z))$ will be continuous as $\mathrm{Im}(z) \to (2k+1)\pi^-$. Therefore, since $\mathrm{Im}(\omega(z) + \ln\omega(z)) = \mathrm{Im}(z)$ for $(2k-1)\pi < \mathrm{Im}(z) < (2k+1)\pi$, we have $\mathcal{K}(\omega + \ln\omega) = \mathcal{K}(z)$ even if $\mathrm{Im}(z) = (2k+1)\pi$ by continuity:

$$\begin{aligned} &\lim_{\mathrm{Im}(z)\to(2k+1)\pi^-} \mathcal{K}(\omega(z) + \ln\omega(z)) \\ &= \lim_{\mathrm{Im}(z)\to(2k+1)\pi^-} \mathcal{K}(z)\,. \end{aligned}$$

Therefore $\mathcal{K}(\omega(z) + \ln\omega(z)) = \mathcal{K}(z)$ unless $\omega(z) < 0$, and $\mathrm{Im}(z) = -i\pi$.

(vi) This now follows immediately.

2.4 Corollary

Define $z(k,\theta) = x + i\cdot(2k+\theta)\pi$. Then $z(k+1,-1) = z(k,1)$ since $x + i\cdot(2k+2-1)\pi = x + i\cdot(2k+1)\pi$, since $\mathcal{K}(x+i\cdot(2k+\theta)\pi) = k$ for $-1 < \theta \le 1$. Since

$$W_k(e^{x+i(2k+\theta)\pi}) = W_k(e^{x+i\pi\theta}) = W_k(e^x(\cos\pi\theta + i\sin\pi\theta))\ ,$$

we have $W_k(e^{x+i\pi\theta}) \to W_k(-e^x + i\cdot 0^+)$ as $\theta \to 1^-$, and

$$\lim_{\theta\to -1^+} W_{\mathcal{K}(z(k+1,\theta))}(e^{z(k+1,\theta)}) = \lim_{\theta\to -1^+} W_{k+1}(e^{x+i(2k+2+\theta)\pi})$$

since $\mathcal{K}(x+i\cdot(2k+2+\theta)\pi) = k+1$ for $-1 < \theta \le 1$. Since $W_{k+1}(e^{x+i\cdot(2(k+1)+\theta\pi)}) = W_{k+1}(e^{x+i\pi\theta}) = W_{k+1}(e^x(\cos\pi\theta + i\sin\pi\theta))$ we have

$$W_{k+1}(e^{x+i\pi\theta}) \to W_{k+1}(-e^x + i\cdot 0^-)$$

as $\theta \to -1^+$. By continuity of ω, then, unless $k=0$ or $k=-1$ and $0 > -e^x > 1$

$$W_k(-e^x + i\cdot 0^+) = W_{k+1}(-e^x + i\cdot 0^-)\ .$$

Alternative (direct) proof of (iv) (a).

Lemma: $W_k(-e^x + i\cdot 0^+) = W_{k+1}(-e^x + i\cdot 0^-)$ unless $-e^{-1} \le -e^x < 0$ and $k=0$.

Proof.

Images of the lines $y = t$, $x = \text{constant} < 0$, are smooth curves under $\omega(x+iy)$, by inspection, except if $\omega(x+iy) < 0$. This is what we have to prove. Can we define e^z as a value on the Riemann Surface for W? Yes, except when $-e^{-1} \le e^z < 0$, by placing it on the sheet with winding number $\mathcal{K}(z)$. This is a bijection between the Riemann Surface for log and for W, except on $-e^{-1} \le e^z < 0$. Once this is done, the cut's images on the Riemann Surface can obviously be moved at will. Since $W_k(-e^x + i\cdot 0^-)$ lies on the other side of the cut, we have equality.

Algebraic properties

- Derivatives and integrals:

$$\frac{d\omega}{dz} = \frac{\omega}{1+\omega}$$

$$\int \omega^n\, dz = \begin{cases} \frac{\omega^{n+1}-1}{n+1} + \omega^n/n & \text{if } n \ne -1 \\ \ln\omega - 1/\omega & \text{if } n = -1 \end{cases}$$

  The derivative formula is valid except on the doubling line and its reflection, when it is valid as a derivative in the real direction only. The integrals can be verified directly by differentiation of both sides. The addition of the constant term $-1/(n+1)$ to the integral of ω^n is a trick due, in the case $\int x^n\,dx$, to W. Kahan. Using this trick, the formula for limiting case $n=-1$ is a simple limit of the formula for $n \ne -1$.

– Series about $z = a$, where $a = \omega_a + \ln \omega_a$: the following (computed by Maple) is the beginning of the series for ω which contains second order Eulerian numbers.

$$
\begin{aligned}
&\omega_a + \frac{\omega_a}{1+\omega_a}(z-a) + \frac{1}{2!}\,\frac{\omega_a}{(1+\omega_a)^3}(z-a)^2 \\
&-\frac{1}{3!}\,\frac{\omega_a\,(2\,\omega_a-1)}{(1+\omega_a)^5}(z-a)^3 + \frac{1}{4!}\,\frac{\omega_a\,(6\,{\omega_a}^2-8\,\omega_a+1)}{(1+\omega_a)^7}(z-a)^4 \\
&-\frac{1}{5!}\,\frac{\omega_a\,(24\,{\omega_a}^3-58\,{\omega_a}^2+22\,\omega_a-1)}{(1+\omega_a)^9}(z-a)^5 \\
&+\frac{1}{6!}\,\frac{\omega_a\,(120\,{\omega_a}^4-444\,{\omega_a}^3+328\,{\omega_a}^2-52\,\omega_a+1)}{(1+\omega_a)^{11}}(z-a)^6 \\
&+O((z-a)^7)
\end{aligned}
$$

The general term is [6]:

$$\omega(z) = \sum_{n\geq 0} \frac{q_n(\omega_a)}{(1+\omega_a)^{2n-1}}\,\frac{(z-a)^n}{n!} \tag{4}$$

where

$$q_n(w) = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1}\,. \tag{5}$$

is defined in terms of second order Eulerian numbers.

– Series about ∞: This series was originally due to de Bruijn, and Comtet identified the coefficients as Stirling numbers.

$$
\begin{aligned}
&\omega \sim z - \ln(z) + \frac{\ln(z)}{z} + \frac{1}{2}\,\frac{\ln(z)\,(\ln(z)-2)}{z^2} \\
&+\frac{1}{6}\,\frac{\ln(z)\,(-9\ln(z)+6+2\ln(z)^2)}{z^3} + \frac{1}{12} \\
&\frac{\ln(z)\,(3\ln(z)^3-22\ln(z)^2+36\ln(z)-12)}{z^4} + \\
&\frac{1}{60}\ln(z)(-125\ln(z)^3+350\ln(z)^2+12\ln(z)^4 \\
&-300\ln(z)+60)/z^5 + O(\frac{1}{z^6})
\end{aligned}
$$

The general term is (translating from the Lambert W results of [2, 7]) $\omega(z) =$

$$z - \ln z + \sum_{\ell\geq 0}\sum_{m\geq 0} c_{\ell m}\frac{\ln^m z}{z^{\ell+m}} \tag{6}$$

where $c_{\ell m} = (-1)^\ell \left[{\ell+m \atop \ell+1}\right]/m!$ is defined in terms of Stirling cycle numbers [8]. This series converges for large enough z, outside the region bounded by the doubling line and its reflection. The proof is unpublished. The series can be rearranged in several ways, following [9] and [6]: $\omega(z) =$

$$z - \ln z + \sum_{n\geq 1} \frac{(-1)^n}{z^n} \sum_{m=1}^{n} \frac{(-1)^m}{m!} \begin{bmatrix} n \\ n-m+1 \end{bmatrix} \ln^m z\,. \tag{7}$$

Using a new variable $\zeta = z/(1+z)$, we get $\omega(z) =$

$$z - \ln z + \sum_{m\geq 1} \frac{\ln^m z}{m! z^m} \sum_{p=0}^{m-1} (-1)^{p+m-1} \zeta^{p+m} \left\{ \begin{matrix} p+m-1 \\ p \end{matrix} \right\}_{\geq 2} \tag{8}$$

where the numbers in curly braces are 2-associated Stirling numbers. Using $L_\tau = \ln(1-\tau) = \ln(1 - \ln z/z)$ and $\eta = \sigma/(1-\tau) = 1/(z(1-\ln z/z)) = 1/(z - \ln z)$, series (83) and (84) from [6] become

$$\begin{aligned} \omega(z) &= z - \ln z - L_\tau \\ &+ \sum_{n\geq 1} (-\eta)^n \sum_{m=1}^{n} (-1)^m \begin{bmatrix} n \\ n-m+1 \end{bmatrix} \frac{L_\tau^m}{m!} \end{aligned} \tag{9}$$

and

$$\begin{aligned} \omega(z) &= z - \ln z - L_\tau \\ &+ \sum_{m\geq 1} \frac{1}{m!} L_\tau^m \eta^m \sum_{p=0}^{m-1} \left\{ \begin{matrix} p+m-1 \\ p \end{matrix} \right\}_{\geq 2} \frac{(-1)^{p+m-1}}{(1+\eta)^{p+m}}\,. \end{aligned} \tag{10}$$

The series converge for large enough real z, though the detailed regions of convergence are not yet settled. Curiously enough (10) is exact at $z = 1$ and at $z = \infty$, and moreover if we truncate it to N terms it agrees with the N term Taylor series expansion at $z = e$ as well, making one think of 'Hermite' interpolation at 1 and at ∞. Convergence is rapid.

– Series about $-\infty$: from the series $W(z) = \sum_{n\geq 1} (-n)^{n-1} z^n/n!$, for $|\exp(z)| < \exp(-1)$ we have

$$\omega(z) = \sum_{n\geq 1} \frac{(-n)^{n-1}}{n!} e^{nz} \tag{11}$$

2.5 Branch point series for $\omega(z)$

The Wright ω function has branch points at $z = -1 \pm i\pi$. The following series obtain. Near $z = -1 + i\pi$,

$$\omega(z) = -\sum_{n\geq 0} a_n \left(i\sqrt{\overline{\overline{2(z+1-i\pi)}}} \right)^n \tag{1}$$

where the double conjugation gives us the correct closure from below on $t + i\pi$ for $t \leq -1$. Near $z = -1 - i\pi$,

$$\omega(z) = -\sum_{n \geq 0} a_n \left(-i \overline{\sqrt{\overline{2(z + 1 + i\pi)}}} \right)^n . \tag{2}$$

In both cases a_n is given by the recurrence relation [10]

$$\begin{aligned} a_0 &= a_1 = 1 \\ a_k &= \frac{1}{(k+1)a_1} \left(a_{k-1} - \sum_{i=2}^{k-1} i a_i a_{k+1-i} \right) . \end{aligned} \tag{3}$$

The derivation of these series from the results of [10] is straightforward, except for the use of $\overline{\sqrt{\overline{z}}}$. We here verify that this construction, which is one of a family of transformations modelled on some used by G.K. Batchelor, gives us the correct closure. We know that $\omega(t + i\pi^-) = W_0(-e^t)$ whilst $\omega(t + i\pi^+) = W_1(-e^t)$, and $\omega(t - i\pi^+) = W_0(-e^t)$ whilst $\omega(t - i\pi^-) = W_{-1}(-e^t)$. Putting $z = t + i\pi^+$ in $\overline{\sqrt{\overline{2(z + 1 - i\pi)}}}$ gives $\overline{\sqrt{\overline{2(t + 1 + i \cdot 0^+)}}}$, for $t \approx -1$. If $t + 1 \geq 0$ then we have no branch cut to cross—this series will be continuous, therefore, along the line $t+1+i\pi$, $t \geq -1$. If $t+1 < 0$, we are on the branch cut. $\overline{t + 1 + i \cdot 0^+}$ is $t+1+i\cdot 0^-$, and arg $\sqrt{2(t + 1 + i \cdot 0^-)} = -\pi/2$. Therefore arg $\overline{\sqrt{2(t + 1 + i \cdot 0^-)}} = +\pi/2$, and this means that the series (2) can be written

$$\omega(z) = -\sum_{n \geq 0} a_n (\rho)^n$$

and by inspection of the signs of the series for $W_{-1}(-e^t)$ and hence $W_{+1}(-e^t)$ just above the branch cut, this is correct. [Here $\rho = \sqrt{-2(t+1)} > 0$.] Next, consider $z = -1 + i\pi^-$. A similar argument leads to the conclusion

$$\omega(z) = -\sum_{n \geq 0} a_n (-\rho)^n$$

which is the series for $W_0(-e^t)$ for $t \approx -1$, because its signs alternate. Consideration of $z = t - i\pi^+$ and $t - i\pi^-$ gives, for $t + 1 < 0$,

$$\begin{aligned} \omega(z) &= -\textstyle\sum_{n \geq 0} a_n (-\rho)^n \quad z = t - i\pi^+ \\ &= -\textstyle\sum_{n \geq 0} a_n \rho^n \quad\quad z = t - i\pi^- \end{aligned}$$

and continuity if $t + 1 \geq 0$.

Remark. The use of $\overline{\sqrt{\overline{(z - a)}}}$ to represent a square root function with a closure different from the CCC closure, as explained by Kahan, is a useful tool in a computer algebra setting. However, it relies on the designers to be sophisticated enough to provide symbolic means of representing (and not over-simplifying) these series, and the users to be sophisticated enough to know that $\overline{\sqrt{\overline{z}}} \neq \sqrt{z}$ on the branch cut.

3 Interpolating $W_k(z)$

Finally, we interpret equation (3) as an interpolation scheme for $W_k(z)$. We note that k need not be an integer in that equation; the geometric interpretation is precisely that of a circular cylinder cutting the Riemann surface for W. Note also that $k = 0$ and $k = -1$ are special, and not interpolated by this scheme.

We deduce that $W_k(z)$ is, in some sense, analytic in k, except if $-\exp(-1) \leq z < 0$ and $k = 0$ or $k = -1$.

$$\begin{aligned} \frac{dW_k(z)}{dk} &= \frac{d}{dk}\omega(\ln z + 2\pi i k) \\ &= 2\pi i \frac{\omega(\ln z + 2\pi i k)}{1 + \omega(\ln z + 2\pi i k)} \ . \end{aligned}$$

By the analytic properties of ω, this derivative is not continuous on $-\exp(-1) \leq z < 0$ at $k = 0$ or $k = -1$. Otherwise, indeed, $W_k(z)$ is analytic in k.

4 Why

Computer algebra is about expressiveness, and simplicity is power. There are an essentially infinite number of applications of the Lambert W function and its cognates.

1. The Lambert W function provides the first example of a function *just* outside the standard body of Risch-like theory: its derivative is rational in x and W, not polynomial. One cannot use the same theorems, but one can hope to use similar methods, to establish its non-elementarity [1].
2. The Lambert W function is the simplest example of a *root of an exponential polynomial*; and exponential polynomials are the next simplest class of functions after polynomials. Computer algebra systems have a real edge over numerical systems (though not everyone knows it) in dealing with polynomials; the next big area will be non-polynomials, starting with exponential polynomials. This is the field of Cylindrical Non-Algebraic Decomposition.
3. The Lambert W function is the first nontrivial example of a multivalued function. The trivial ones (ln and the arc trig functions) have branching behaviour so simple that it doesn't even need a notation: we can say $\ln(z)+2\pi i k$ and not have to invent a new notation $\ln_k(z)$ to do so (though in fact we *have* introduced and used this notation—one can't use $\log_k$ because the "log to the base k" interpretation would get in the way—for conciseness and as the thin entering point of the wedge for more complicated functions). The multivalued nature of W "stress tests" naming conventions, numerics on branches, computer-aided analysis, and the results of series computation. Right now, Maple knows the series for $W_0(z)$ about the branch point $z = -\exp(-1)$, but it doesn't know the series for $W_{-1}(z)$ or $W_1(z)$ about

the same point, even though these series were all introduced in the same paper [4]. We think that this is because the series are defined *piecewise*: for W_{-1} and W_1, the series about the branch point have to deal with the fact that the range is split by the branch cut, and so the series are (radically) different if $\mathrm{Im}(z) > 0$ or $\mathrm{Im}(z) < 0$; each branch of W has both a Puiseux series and a Taylor series—about the same point! But different series apply above and below the branch cut. This remarkable behaviour puts a significant stress on the ability of `series` to express its answer to the question `series(LambertW(-1,x),x=-exp(-1))` (which it currently refuses to answer).

4.1 Why Invent the Wright ω Function?

It is certain that for some applications, just the ordinary Lambert W function will be superior—this new function cannot supplant the old. Bill Gosper did not succeed in introducing his cognate of W (which he jokingly called "the Dilbert Lambda Function"); Don Knuth has so far been unable to get action on our promise to him to introduce the TreeT function into Maple ($T(x) = -W(-x)$, and this is more convenient for combinatorial applications). So why should we bother with a new one?

In equation (1) we give the definition of the Wright ω function, in terms of W and one new function, the unwinding number $\mathcal{K}(z)$, which we will be needing anyway. So why not just use the right hand side of the definition and not bother with a notation?

1. W is multivalued, but ω is single-valued.
2. Numerically evaluating $\omega(z)$ for large z by way of the definition (1) is like driving from the south of London to the north of London via Waterloo[3]: it's possible (unless there's freezing rain) but unless you have a reason to be in Waterloo, it's probably better to go directly. Less metaphorically, taking $\exp(z)$ for large z gives a significant risk of overflow, and a significant restriction on the numerical range of z that we can do the computation for; but W is like ln, and in some sense just undoes the exponentiation, making it wasted effort in any case. The asymptotics are that $\omega(z) \sim z - \ln z + \cdots$; so we see just how wasted. This is *not a theoretical consideration*: Jon Borwein has had to implement his version of ω precisely to avoid this overflow difficulty in his convex optimization problems.
3. Numerically evaluating `omega( -0.9 + I*Pi )` by way of the formula uncovers a subtle difficulty: because `ceil( (Im(z)-Pi)/(2*Pi) )` will do some symbolic processing, it will compute $\mathcal{K}(z)$ exactly right, and cancel the symbolic `Pi`. But `exp(-0.9+I*Pi)` is left alone, until the user calls `evalf`. Then something awful happens: at 10 Digits, `Pi` rounds to something larger than

[3] London, Ontario via Waterloo, Ontario, of course. That sentence reads quite differently if you think of train travel in London, England, for example (thanks to Arthur Norman for pointing that out)

π; this then gives us a *negative imaginary part* on the order of roundoff in the result of the call to `exp`. This is all explainable in terms of the Maple model of floating-point arithmetic, but it's a disaster nonetheless—one made visible by the next step, the computation of $W_0(x - i \cdot \varepsilon)$, which is *on the wrong side of the branch cut*. The numerical value of $W_0(x - i \cdot \varepsilon)$ is not at all close to the value of $W_0(x + i \cdot \varepsilon)$, and this discontinuity is *spurious*. The ω function is *continuous* at this point. So: we should have a separate routine for the numerical evaluation of ω that guarantees that we get continuity (where ω is continuous), because the definition combines *discontinuous* functions in such a way that their discontinuities (mostly) cancel.

There are other advantages to using the Wright ω function directly.

1. In addition to being single-valued, ω is continuous (indeed analytic) for all z not on the two half-lines $z = t \pm i\pi$ for $t \leq -1$. It is discontinuous across these lines.
2. The Wright ω function has a simpler Taylor series than the Lambert W function does. Indeed, it is the series for the Wright ω function that leads to nearly all the series given in [6].
3. The fabulously simple equation $W_k(z) = \omega(\ln z + 2\pi ik) = \omega(\ln_k z)$ explains the branching behaviour of W perfectly, once we understand the branching behaviour of ω.
4. The solution of the equation $y + \ln y = z$ is given by

$$y = \begin{cases} \omega(z) & z \neq t \pm i\pi, t \leq -1 \\ \omega(z), \omega(z - 2\pi i) & z = t + i\pi, t \leq -1 \\ \text{nonesuch} & z = t - i\pi, t \leq -1 \end{cases} \tag{12}$$

   The paper [11] seems to be the first to use this fact.

What are the disadvantages? Well, the principal one is that the counting applications depend on the use of W (or, rather, TreeT) as a generating function. There, the series at the origin is what is important. With this transformation, we have moved this point to $-\infty$. The series are still there—just less convenient. And, that is what introducing this function is all about: convenience. We will need to have all of these functions around—well, certainly TreeT, but probably not Dilbert Lambda. Even Bill Gosper has mostly given up on that one.

5 Concluding Remarks

This paper presents a number of mathematical results describing the properties of the function $\omega(z)$. These results have some intrinsic mathematical interest, and they are written here for the first time, and so in a technical sense the paper contains novel results. However, the results are really interesting only because:

1. Without symbolic computation making the function's definition, simplification rules, and numerical evaluation widely available, the function is merely arcane

2. Discontinuity (along the branch cuts) is especially visible, and nontrivial, in this function. Therefore it will make a good test case for reasoning about complex-valued expressions.
3. The methods used to prove properties of ω are essentially old-fashioned mathematics, not commonly seen in standard curricula, and may potentially be automated. This is in the spirit of [3] and represents a potentially interesting direction for future research.

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