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# Stress moments of nearly touching spheres in low Reynolds number flow

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## 1. Introduction

The forces and couples acting on two nearly touching spherical particles in low Reynolds number flow have been studied by O'Neill & Majumdar [11] and Jeffrey & Onishi [8] using a method developed by O'Neill & Stewartson [12], and by Goldman, Cox and Brenner [5]. The force acting on a sphere is the zeroth moment of the stress on the sphere's surface and the couple is the antisymmetric part of the first moment. The symmetric part of the first moment (the "stresslet" defined below) plays an important role in studies of the properties of suspensions [1], and recently numerical data has been published for the stresslets of two equal spheres by Kim & Mifflin [10] and by Jeffrey [7]. The data suggests that the stresslets are singular when the spheres touch, just as the force and couple are, and this is verified here; the singular behaviour of the stresslets is calculated by extending the work of Jeffrey & Onishi [8].

The stresslet of a rigid particle is defined to be the symmetric part of the first moment of the surface stress, and it measures the influence that the particle has on the ambient flow field through the fact that a rigid particle does not deform with the flow [1]. Explicitly

$$\boldsymbol{S} = -\int_A \left[\tfrac{1}{2}(\boldsymbol{x}'\boldsymbol{\sigma}\cdot\boldsymbol{n} + \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\boldsymbol{x}') - \tfrac{1}{3}\boldsymbol{I}\,\boldsymbol{x}'\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\right] \mathrm{d}A\,. \tag{1.1}$$

Since the couple acting on a particle is equivalent to the antisymmetric part of the first moment of the surface stress, the couple and stresslet are a natural pair of quantities to calculate.

We study the same flow problems as in earlier papers [8, 11]: firstly, one sphere at rest and one rotating about an axis perpendicular to their line of centres, and secondly, one sphere at rest and one translating perpendicular to the line of centres. The method for solving these problems is that of matched asymptotic expansions, and the complete program of calculating inner and outer expansions and matching them was carried through in the first papers [3, 12]. Later papers, however, have calculated only the solution in the gap between the particles, using lubrication theory approximations, and from that solution found the singular interaction terms. This is possible because the flow problem is somewhat unusual from the point of view of the general theory of matched asymptotic expansions. The solution in the gap appears to be independent of the flow outside the gap. Thus although in other problems it would not be possible to pursue the inner solution to high order without also solving for the outer solution, here it is. The main reason for finding only the singular terms is that the interactions between the spheres are wanted for all separations, not just when they are close to touching, and so the asymptotic results must sooner or later be combined with other solutions. It is most efficient to obtain the singular terms by the present method and the non-singular terms by other methods, such as twin multipole expansions or collocation [9, 10].

## 2. Governing equations

We consider a moving sphere with radius $a$ and a stationary sphere with radius $b$. We take Cartesian axes $(ax, ay, az)$ with the $z$ axis along the line of centres of the spheres and the origin on the surface of the stationary sphere. We also define cylindrical coordinates $(ar, \theta, az)$ with $\theta = 0$ along the $x$ axis. Figure 1 shows a cross-section of the $xz$ plane. The gap between the spheres is the small quantity $a\varepsilon$ and therefore the sphere surfaces are given by

$$(z - 1 - \varepsilon)^2 + r^2 = 1 \quad \text{and} \quad (z + b/a)^2 + r^2 = b^2/a^2 . \tag{2.1, 2.2}$$

The flow around the spheres obeys the Stokes equations

$$\nabla p = \mu \nabla^2 u \quad \text{and} \quad \nabla \cdot u = 0 .$$

For the first problem, the moving sphere has angular velocity $\Omega j$, from which we obtain a velocity scale $\mathscr{U} = \Omega a$. Using $\mathscr{U}$, we can write the boundary conditions on the moving sphere in terms of the cylindrical velocity components $(u, v, w)$, as

$$u = \mathscr{U}(-\cos\theta\cos\phi, \sin\theta\cos\phi, -\cos\theta\sin\phi) .$$

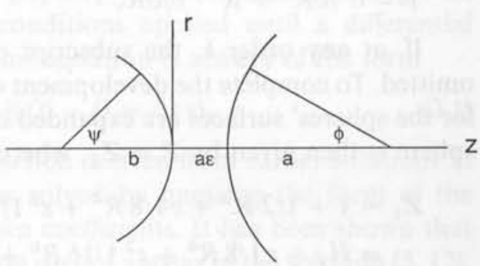


Figure 1
The $xz$ plane, showing the axes of the cylindrical coordinate system and the angles $\phi$ and $\psi$.

The angle $\phi$ is shown in Fig. 1. For the second problem, the sphere translates with velocity $u = \mathscr{U} i$ and the boundary condition is

$$u = \mathscr{U}(\cos\theta, -\sin\theta, 0).$$

On the stationary sphere, $u = 0$ for both problems. The established asymptotic scaling in the gap uses the stretched variables $Z$ and $R$ defined by

$$Z = z/\varepsilon \quad \text{and} \quad R = r/\varepsilon^{1/2}.$$

By combining the linearity of the Stokes equations with the boundary conditions and the asymptotic scaling, one can show that the velocity components and the pressure can be written, for both problems, as [11]

$$u = \mathscr{U}(U\cos\theta, V\sin\theta, W\cos\theta), \tag{2.3}$$

and

$$p = \mu \mathscr{U} P \cos\theta/a, \tag{2.4}$$

where

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + O(\varepsilon^3),$$
$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + O(\varepsilon^3),$$
$$W = \varepsilon^{1/2}(W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + O(\varepsilon^3)),$$

and

$$P = \varepsilon^{-3/2}(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + O(\varepsilon^3)).$$

Substituting the above expressions into the Stokes equations, we obtain the following equations governing the order $k$ quantities.

$$\partial P_k/\partial Z = \partial^2 W_{k-1}/\partial Z^2 + Y W_{k-2} - W_{k-2}/R^2, \tag{2.5}$$

$$\partial P_k/\partial R = \partial^2 U_k/\partial Z^2 + Y U_{k-1} - 2(U_{k-1} + V_{k-1})/R^2, \tag{2.6}$$

$$P_k/R = \partial^2 V_k/\partial Z^2 + Y V_{k-1} - 2(U_{k-1} + V_{k-1})/R^2, \tag{2.7}$$

$$\partial W_k/\partial Z + (U_k + V_k)/R + \partial U_k/\partial R = 0, \tag{2.8}$$

where the operator $Y$ is

$$Y = \partial^2/\partial R^2 + R^{-1}\partial/\partial R. \tag{2.9}$$

If, at any order $k$, the subscript of a term is negative, then that term is omitted. To complete the development of the asymptotic problem, the equations for the spheres' surfaces are expanded in powers of $\varepsilon$. The surface of the moving sphere is then given by $Z = Z_1$, where

$$\begin{aligned} Z_1 &= 1 + 1/2\,R^2 + \varepsilon\,1/8\,R^4 + \varepsilon^2\,1/16\,R^6 + O(\varepsilon^3), \\ &= H_1 + \varepsilon\,1/8\,R^4 + \varepsilon^2\,1/16\,R^6 + O(\varepsilon^3). \end{aligned} \tag{2.10}$$

On the stationary sphere, $Z = Z_2$, where

$$Z_2 = 1/2\kappa R^2 + \varepsilon\, 1/8\kappa^3 R^4 + \varepsilon^2\, 1/16\kappa^5 R^6 + O(\varepsilon^3),$$
$$= H_2 + \varepsilon\, 1/8\kappa^3 R^4 + \varepsilon^2\, 1/16\kappa^5 R^6 + O(\varepsilon^3), \tag{2.11}$$

where we have introduced $\kappa = -a/b$. We use $\kappa$, instead of the $\lambda$ used in earlier papers, so that $\lambda$ can be used in the sense of Jeffrey & Onishi [9], i.e.

$$\lambda = b/a = -1/\kappa.$$

## 3. Rotating sphere

On the rotating sphere the boundary conditions are

$$U(Z_1, R) = -V(Z_1, R) = -1 + \varepsilon(1 - Z_1) \quad\text{and}\quad W(Z_1, R) = -\varepsilon^{1/2} R.$$

Substituting (2.10) for $Z_1$ and expanding each velocity component as a Taylor series we obtain the boundary conditions that apply at each order in terms of functions found at previous orders.

$$U_0(H_1, R) = -V_0(H_1, R) = -1, \quad W_0(H_1, R) = -R.$$
$$U_1(H_1, R) = 1 - H_1 - 1/8\, R^4\, \partial U_0/\partial R,$$
$$V_1(H_1, R) = -1 + H_1 - 1/8\, R^4\, \partial V_0/\partial R,$$
$$W_1(H_1, R) = -1/8\, R^4\, \partial W_0/\partial R.$$

In the equations, the derivatives are evaluated at $Z = H_1$. On the other sphere, we can shift the zero velocity condition to $Z = H_2$ in the same way.

$$U_0(H_2, R) = V_0(H_2, R) = W_0(H_2, R) = 0.$$
$$U_1(H_2, R) = -1/8\, R^4 \partial U_0/\partial R,$$
$$V_1(H_2, R) = -1/8\, R^4\, \partial V_0/\partial R,$$
$$W_1(H_2, R) = -1/8\, R^4\, \partial W_0/\partial R.$$

The equations were integrated using the symbolic mathematics package MAPLE, following the method of earlier papers [4]. Each equation is integrated with respect to $Z$ and the boundary conditions applied until a differential equation is obtained for the pressure. This equation is always of the form

$$R^2 Q'' + [R + 3(1-\kappa) R^3/(H_1 - H_2)]\, Q' - Q = f_k(x), \tag{3.1}$$

where the right-hand side is a known function derived from earlier solutions in the asymptotic scheme. The equation is solved by guessing the form of the particular integral and adjusting unknown coefficients. It has been shown that the solutions of the homogeneous equation do not appear in the solution [5, 12].

To calculate the stresslet, we first rewrite the components of the stress tensor, separating the dependence on $\theta$ in the way it was done for velocity and pressure.

$$\sigma_{rr} = \mu\Omega\cos\theta(-P + 2\,\partial U/\partial r) = \mu\Omega\cos\theta\,\hat\sigma_{rr}, \tag{3.2}$$

$$\sigma_{rz} = \mu\Omega\cos\theta(\partial U/\partial z + \partial W/\partial r) = \mu\Omega\cos\theta\,\hat\sigma_{rz}, \tag{3.3}$$

$$\sigma_{\theta z} = \mu\Omega\sin\theta(-W/r + \partial V/\partial z) = \mu\Omega\sin\theta\,\hat\sigma_{\theta z}, \tag{3.4}$$

$$\sigma_{\theta r} = \mu\Omega\sin\theta(\partial V/\partial r - V/r - U/r) = \mu\Omega\sin\theta\,\hat\sigma_{\theta r}, \tag{3.5}$$

$$\sigma_{zz} = \mu\Omega\cos\theta(-P + 2\,\partial W/\partial z) = \mu\Omega\cos\theta\,\hat\sigma_{zz}. \tag{3.6}$$

We reduce the stresslet to the scalar function $Y_{11}^H$ by extending the established notation to unequal spheres [7, 9, 10]. The stresslet on the moving sphere is given by $S = 8\pi a^3\, Y_{11}^H(ik + ki)\Omega$. Taking the $xz$ component of (1.1), we can integrate over $\theta$ to be left with an integral over $\phi$, where $\phi$ is defined in Fig. 1. Thus

$$Y_{11}^H = \tfrac{1}{16}\int\{-\cos\phi\,[\hat\sigma_{rr}\sin\phi - \hat\sigma_{rz}\cos\phi - \hat\sigma_{\theta r}\sin\phi + \hat\sigma_{\theta z}\cos\phi] + \sin\phi\,[\hat\sigma_{zr}\sin\phi - \hat\sigma_{zz}\cos\phi]\}\sin\phi\,d\phi. \tag{3.7}$$

The stresslet of the stationary sphere is $S = \mu\pi(a+b)^3\, Y_{21}^H(ik + ki)\Omega$. The integral for $Y_{21}^H$ is

$$Y_{21}^H = \frac{\lambda^3}{2(1+\lambda)^3}\int\{\cos\psi\,[\hat\sigma_{rr}\sin\psi + \hat\sigma_{rz}\cos\psi - \hat\sigma_{\theta r}\sin\psi - \hat\sigma_{\theta z}\cos\psi] + \sin\psi\,[\hat\sigma_{zr}\sin\psi + \hat\sigma_{zz}\cos\psi]\}\sin\psi\,d\psi, \tag{3.8}$$

where $\psi$ is the angle shown on Fig. 1.

To obtain the singular behaviour of $Y_{11}^H$ and $Y_{21}^H$, we expand the integrand of the above integrals for large $R$ and obtain

$$\text{integrand} \sim a_0 R + a_1/R + O(R^{-3}),$$

from which the singular behaviour follows as [10]

$$Y_{\alpha\beta}^H \sim \tfrac{1}{2}a_1 \ln\varepsilon^{-1} + O(1).$$

This leads to the results

$$Y_{11}^H \sim \frac{1}{10}\frac{2\lambda - \lambda^2}{(1+\lambda)^2}\ln\varepsilon^{-1} + \frac{1}{250}\frac{16 - 61\lambda + 180\lambda^2 + 2\lambda^3}{(1+\lambda)^3}\varepsilon\ln\varepsilon^{-1} + O(1), \tag{3.9}$$

and

$$Y_{21}^H \sim \frac{2}{5}\frac{7\lambda^2 + \lambda^3}{(1+\lambda)^5}\ln\varepsilon^{-1} + \frac{2}{125}\frac{221\lambda - 185\lambda^2 + 147\lambda^3 + 43\lambda^4}{(1+\lambda)^6}\varepsilon\ln\varepsilon^{-1} + O(1). \tag{3.10}$$

The corresponding results for $Y_{12}^H$ and $Y_{22}^H$ can be found from interchanging the labels of the spheres [9]. Thus

$$Y_{22}^H \sim \frac{1}{10} \frac{2\lambda - 1}{(1+\lambda)^2} \ln \varepsilon^{-1} + \frac{1}{250} \frac{16\lambda^3 - 61\lambda^2 + 180\lambda + 2}{\lambda(1+\lambda)^3} \varepsilon \ln \varepsilon^{-1} + O(1), \tag{3.11}$$

and

$$Y_{12}^H \sim \frac{2}{5} \frac{7\lambda^3 + \lambda^2}{(1+\lambda)^5} \ln \varepsilon^{-1} + \frac{2}{125} \frac{221\lambda^4 - 185\lambda^3 + 147\lambda^2 + 43\lambda}{(1+\lambda)^6} \varepsilon \ln \varepsilon^{-1} + O(1). \tag{3.12}$$

For the case of equal spheres ($\lambda = 1$), the expressions above all agree with the expressions deduced from studying the coefficients of infinite series expressions for the same quantities [7].

## 4. Translating sphere

When the moving sphere translates, the boundary conditions are

$$U = -V = 1 \quad \text{and} \quad W = 0.$$

On the stationary sphere the velocities are zero. The solution scheme proceeds as in the previous section. The stresslets are in this case expressed using the functions $Y_{11}^G$ and $Y_{21}^G$. For the moving sphere

$$S = -4\pi a^2 Y_{11}^G (ik + ki) U,$$

and for the stationary sphere,

$$S = -\pi (a+b)^2 Y_{21}^G (ik + ki) U.$$

The minus sign comes from the fact that the vector used by Kim and Mifflin [10] to define $Y_{\alpha\beta}^G$ is $-k$. The integrals are the same as those for $Y_{\alpha\beta}^H$ except that the one in (3.7) is multiplied by 2 and the one in (3.8) is multiplied by $1 + \lambda$. Thus

$$Y_{11}^G = \frac{1}{10} \frac{4\lambda - \lambda^2 + 7\lambda^3}{(1+\lambda)^3} \ln \varepsilon^{-1} + \frac{1}{250} \frac{32 - 179\lambda + 532\lambda^2 - 356\lambda^3 + 221\lambda^4}{(1+\lambda)^4} \varepsilon \ln \varepsilon^{-1} + O(1), \tag{4.1}$$

and

$$Y_{21}^G = \frac{2}{5} \frac{7\lambda^2 - \lambda^3 + 4\lambda^4}{(1+\lambda)^5} \ln \varepsilon^{-1} + \frac{2}{125} \frac{221\lambda - 356\lambda^2 + 532\lambda^3 - 179\lambda^4 + 32\lambda^5}{(1+\lambda)^6} \varepsilon \ln \varepsilon^{-1} + O(1). \tag{4.2}$$

By relabelling the spheres, we can show from these results, that

$$4\,Y_{11}^{G} + (1+\lambda)^2\,Y_{12}^{G} = O(1). \tag{4.3}$$

In other words, the singular terms in the two functions are equal (but the functions are otherwise different). This can be understood as follows. If the two spheres both have the same translational velocity $U\,i$, the stresslet of sphere $a$ will be

$$S = \mu(4\pi a^2\,Y_{11}^{G} + \pi(a+b)^2\,Y_{12}^{G})\,U(ik + ki).$$

It has been pointed out before [8] that if there is no shearing of the fluid in the gap, as in this case where both surfaces move with the same velocity, there can be no singular behaviour. Thus the singular terms in the two functions must cancel in the way they do. In the same way, we find

$$4\lambda^2\,Y_{22}^{G} + (1+\lambda)^2\,Y_{21}^{G} = 0(1). \tag{4.4}$$

For $\lambda = 1$, the singular terms again agree with those deduced by studying infinite series [7].

## 5. Application of reciprocal theorem

Some remarkable relations can be found between the functions $Y_{\alpha\beta}^{G}$ and $Y_{\alpha\beta}^{H}$ studied here and the functions $Y_{\alpha\beta}^{A}$ and $Y_{\alpha\beta}^{B}$ studied in Jeffrey & Onishi [9]. Here $Y_{\alpha\beta}^{G}$ and $Y_{\alpha\beta}^{H}$ where found by calculating the stresslets of moving spheres placed in an ambient flow that is at rest far from the spheres. The reciprocal theorem, however, requires that the same functions appear in expressions for the forces and couples on two spheres placed in an ambient rate-of-strain field [2, 10, 13]. Thus if two spheres are held stationary in a rate-of-strain field $E(ik + ki)$, using the same coordinates as above, the force and couple on the sphere of radius $a$ are

$$F = 6\pi a\mu\,Y_{11}^{A}\,[-a(1+\varepsilon)E]\,i + 3\pi(a+b)\mu\,Y_{12}^{A}\,bEi \\ + [4\pi a^2\mu\,Y_{11}^{G} + \pi(a+b)^2\mu\,Y_{21}^{G}]\,2Ei, \tag{5.1}$$

and

$$L = 4\pi a^2\mu\,Y_{11}^{B}\,[-a(1+\varepsilon)E]j + \pi(a+b)^2\mu\,Y_{12}^{B}\,bEj \\ - [8\pi a^3\mu\,Y_{11}^{H} + \pi(a+b)^3\mu\,Y_{21}^{H}]\,2Ej. \tag{5.2}$$

We now argue that since the spheres are not in relative motion, the force and couple cannot be singular in the gap width $\varepsilon$. Therefore the $\ln\varepsilon$ and $\varepsilon\ln\varepsilon$ singularities in the $Y_{\alpha\beta}^{A}$ must cancel those in the $Y_{\alpha\beta}^{G}$. Similarly the singular terms in $Y^B_{\alpha\beta}$ must cancel those in $Y^H_{\alpha\beta}$. We conclude that the combinations

$$-6\,Y^A_{11}(1+\varepsilon) + 3(1+\lambda)\lambda\, Y^A_{12} + 8\,Y^G_{11} + 2(1+\lambda)^2\, Y^G_{21} \tag{5.3}$$

and

$$4\,Y^B_{11}(1+\varepsilon) - (1+\lambda)^2\lambda\, Y^B_{12} + 16\,Y^H_{11} + 2(1+\lambda)^3\, Y^H_{21} \tag{5.4}$$

should be independent of $\ln\varepsilon$ and $\varepsilon\ln\varepsilon$. They are. These tests provide further validation of the calculations performed here.

For the special case $\lambda \to \infty$, the results of Goldman, Cox and Brenner [6] can be used as a check. They calculated the force and couple acting on a sphere in a shear flow next to a wall. They used the reciprocal theorem to equate these quantities to integrals that are stresslet integrals, although they did not name the integrals specifically. Converting their results to the notation used here, we get

$$Y^G_{11} = -(3/2)(F^{t*} + F^{r*}/2 + F^{s*}) = (7/10)\log\varepsilon^{-1} - 0.923\,.$$

$$Y^H_{11} = -T^{t*} - T^{r*}/2 - T^{s*}/2 = -(1/10)\log\varepsilon^{-1} + 0.0916\,.$$

## 6. Numerical results

The functions $Y^G_{\alpha\beta}$ and $Y^H_{\alpha\beta}$ have been calculated numerically for the case of equal spheres [7, 10]. We show here that the infinite series used in Jeffrey [7] can be made to converge much faster using the results obtained above. The method is an improved version of that used by Jeffrey & Onishi [9]. The series for $Y^G_{11}$ and $Y^G_{12}$ can be written [7]

$$Y^G_{11} = \tfrac{3}{4}\sum_{\substack{p=1\\ p\ \text{odd}}} P_{2p}\,t^p\,, \tag{6.1}$$

and

$$Y^G_{12} = -\tfrac{3}{4}\sum_{\substack{p=2\\ p\ \text{even}}} P_{2p}\,t^p\,, \tag{6.2}$$

where the $P_{2p}$ are numerical coefficients, 350 of which have been calculated. The variable $t$ is given in the present notation by

$$t = a/(\text{distance between sphere centres}) = 1/(2+\varepsilon)\,.$$

Since the functions are singular at $\varepsilon = 0$, the series representations of these functions must diverge at $t = 1/2$, and be slowly convergent near there. Since we know that the singularity is of the type $\ln\varepsilon$, we expect that the coefficients will be asymptotic to $2^p/p$, and this has been verified.

To improve the convergence, we proceed as follows. We consider the identity

$$g_2 \ln \frac{1+2t}{1-2t} + g_3 \left(\frac{1}{4t^2} - 1\right) \ln \frac{1+2t}{1-2t}$$
$$= g_3/t + \sum_{\substack{p=1 \\ p\,\mathrm{odd}}}^{\infty} \left\{ 2 g_2 \frac{2^p}{p} - 4 g_3 \frac{2^p}{p(p+2)} \right\} t^p . \tag{6.3}$$

We note that the series contain only odd powers of $t$, like $Y_{11}^G$. We now substitute $t = 1/(2+\varepsilon)$ into the left-hand side and expand for small $\varepsilon$. Thus

$$g_2 \ln \frac{1+2t}{1-2t} + g_3 \left(\frac{1}{4t^2} - 1\right) \ln \frac{1+2t}{1-2t}$$
$$= g_2 \ln \varepsilon^{-1} + g_2 \ln 4 + g_3\, \varepsilon \ln \varepsilon^{-1} + O(\varepsilon) .$$

By evaluating (4.1) when $\lambda = 1$, we can select $g_2$ and $g_3$ to capture the asymptotic behaviour of $Y_{11}^G$ by adding the left side of (6.3) to (6.1) and subtracting the right side. Thus

$$Y_{11}^G = \frac{1}{8} \ln \frac{1+2t}{1-2t} + \frac{1}{16} \left(\frac{1}{4t^2} - 1\right) \ln \frac{1+2t}{1-2t}$$
$$- 1/(16t) + \sum_{\substack{p=1 \\ p\,\mathrm{odd}}}^{\infty} \left\{ \frac{3}{4} P_{2p} - \frac{2^p}{4p} + \frac{2^p}{4p(p+2)} \right\} t^p . \tag{6.4}$$

The improvement in the convergence of the infinite series can be seen in Table 1, where the series is summed to 100, 200 and 300 terms for the value $t = 1/(2.01)$. As each singularity is taken into account explicitly, the convergence improves.

Table 1
Convergence properties of series for $Y_{\alpha\beta}^G$ at $\varepsilon = 0.01$.

| | $Y_{11}^G$ | | | $Y_{12}^G$ | | |
|---|---|---|---|---|---|---|
| $n$ | First Form | Second Form | Third Form | First Form | Second Form | Third Form |
| 100 | 0.36921 | 0.43937 | 0.43896 | −0.40819 | −0.47760 | −0.47719 |
| 200 | 0.41159 | 0.43912 | 0.43903 | −0.45004 | −0.47735 | −0.47725 |
| 300 | 0.42649 | 0.43906 | 0.43903 | −0.46481 | −0.47729 | −0.47726 |
| 350 | 0.43031 | 0.43905 | 0.43903 | −0.46860 | −0.47727 | −0.47726 |

The function $Y_{12}^G$ is treated in the same way. This time we use the identity

$$-g_2 \ln(1-4t^2) - g_3\left(\frac{1}{4t^2}-1\right)\ln(1-4t^2)$$

$$= g_3 + \sum_{\substack{p=2\\ p\text{ even}}}^{\infty} \left\{2g_2\frac{2^p}{p} - 4g_3\frac{2^p}{p(p+2)}\right\} t^p. \tag{6.5}$$

Expanding the left-hand side for small $\varepsilon$, we obtain

$$-g_2 \ln(1-4t^2) - g_3\left(\frac{1}{4t^2}-1\right)\ln(1-4t^2)$$

$$= g_2 \ln\varepsilon^{-1} + g_3\,\varepsilon\ln\varepsilon^{-1} + O(\varepsilon).$$

This leads to an improved representation of $Y_{12}^G$ in the form

$$Y_{12}^G = (1/8)\ln(1-4t^2) + (1/16)\left(\frac{1}{4t^2}-1\right)\ln(1-4t^2)$$

$$-g_3 - \sum_{\substack{p=2\\ p\text{ even}}}^{\infty} \left\{\frac{3}{4}P_{2p} - \frac{2^p}{4p} + \frac{2^p}{4p(p+2)}\right\} t^p. \tag{6.6}$$

Again in Table 1, the improvement in the convergence can be seen. Similar expressions can be written down for $Y_{11}^H$ and $Y_{12}^H$, with the only difference that the even powers of $t$ now go with $Y_{11}^H$. The numerical improvement can be seen in Table 2.

Table 2
Convergence properties of series for $Y_{\alpha\beta}^H$ at $\varepsilon = 0.01$.

| | $Y_{11}^H$ | | | $Y_{12}^H$ | | |
|---|---|---|---|---|---|---|
| $n$ | First Form | Second Form | Third Form | First Form | Second Form | Third Form |
| 100 | 0.03120 | 0.04508 | 0.04463 | 0.37834 | 0.43451 | 0.43413 |
| 200 | 0.03925 | 0.04471 | 0.04461 | 0.41216 | 0.43419 | 0.43410 |
| 300 | 0.04215 | 0.04464 | 0.04461 | 0.42408 | 0.43413 | 0.43411 |
| 350 | 0.04289 | 0.04463 | 0.04461 | 0.42713 | 0.43412 | 0.43411 |

To complete our study of the asymptotic forms, we need the $O(1)$ and $O(\varepsilon)$ terms in the expression for the functions. The natural extension of earlier notation is

$$Y_{11}^G = g_2\ln\varepsilon^{-1} + G_{11}^Y + g_3\,\varepsilon\ln\varepsilon^{-1} + O(\varepsilon),$$

$$Y_{12}^G = -g_2\ln\varepsilon^{-1} + G_{12}^Y - g_3\,\varepsilon\ln\varepsilon^{-1} + O(\varepsilon).$$

By expanding (6.4) for small $\varepsilon$ and then setting $\varepsilon = 0$ and $t = 1/2$, we obtain

$$G_{11}^{Y} = (1/8)\ln 4 - 1/8 + \sum_{\substack{p=1 \\ p\,\mathrm{odd}}}^{\infty} \left\{ \frac{3}{4} 2^{-p} P_{2p} - \frac{1}{4p} + \frac{1}{4p(p+2)} \right\}, \tag{6.7}$$

and

$$G_{12}^{Y} = -(1/16) - \sum_{\substack{p=2 \\ p\,\mathrm{even}}}^{\infty} \left\{ \frac{3}{4} 2^{-p} P_{2p} - \frac{1}{4p} + \frac{1}{4p(p+2)} \right\}. \tag{6.8}$$

The $O(\varepsilon)$ terms are found by fitting data for $\varepsilon \leqq 0.02$. We deduce the following results.

$$Y_{11}^{G} = \tfrac{1}{8}\ln\varepsilon^{-1} - 0.1411 + \tfrac{1}{16}\varepsilon\ln\varepsilon^{-1} + 0.16\varepsilon + 0(\varepsilon^2\ln\varepsilon), \tag{6.9}$$

$$Y_{12}^{G} = -\tfrac{1}{8}\ln\varepsilon^{-1} + 0.1025 - \tfrac{1}{16}\varepsilon\ln\varepsilon^{-1} - 0.12\varepsilon + 0(\varepsilon^2\ln\varepsilon), \tag{6.10}$$

$$Y_{11}^{H} = \tfrac{1}{40}\ln\varepsilon^{-1} - 0.0741 + \tfrac{137}{2000}\varepsilon\ln\varepsilon^{-1} + 0.04\varepsilon + 0(\varepsilon^2\ln\varepsilon), \tag{6.11}$$

$$Y_{12}^{H} = \tfrac{1}{10}\ln\varepsilon^{-1} - 0.0294 + \tfrac{113}{2000}\varepsilon\ln\varepsilon^{-1} + 0.04\varepsilon + 0(\varepsilon^2\ln\varepsilon). \tag{6.12}$$

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## References

[1] G. K. Batchelor, J. Fluid Mech. *41*, 545 (1970).
[2] H. Brenner and M. E. O'Neill, Chem. Engng. Sci. *27*, 1421 (1972).
[3] M. D. A. Cooley and M. E. O'Neill, J. Inst. Maths. Applics. *4*, 163 (1967).
[4] R. M. Corless and D. J. Jeffrey, J. Symb. Comp. (submitted).
[5] A. J. Goldman, R. G. Cox and H. Brenner, Chem. Eng. Sci. *22*, 637 (1967).
[6] A. J. Goldman, R. G. Cox and H. Brenner, Chem. Eng. Sci. *22*, 653 (1967).
[7] D. J. Jeffrey, Utilitas Mathematica (in press).
[8] D. J. Jeffrey and Y. Onishi, Z. angew. Math. Phys. *35*, 634 (1984).
[9] D. J. Jeffrey and Y. Onishi, J. Fluid Mech. *139*, 261 (1984).
[10] S. Kim and R. T. Mifflin, Phys. Fluids, *28*, 2033 (1985).
[11] M. E. O'Neill and S. R. Majumdar, Z. angew. Math. Phys. *21*, 180 (1972).
[12] M. E. O'Neill and K. Stewartson, J. Fluid Mech. *27*, 705 (1967).
[13] R. Schmitz & B. U. Felderhof, Physica, *113A*, 103 (1982).

## Abstract

If two spheres are nearly touching, and the flow around them is governed by the Stokes equations, the integral moments of the surface stress are singular functions of the gap width. The method used previously to calculate the singular terms in the zeroth moment (the force) and the antisymmetric first moment (the couple) is extended here to calculate the singular terms in the symmetric first moment (the stresslet) for motions perpendicular to the line of centres. It is shown that the reciprocal theorem requires unexpected relations between the newly found singularities and ones found previously. It is also shown that the singular terms can be used to improve the rate of convergence of series expressions for the stresslets. The series expressions then become valid for all separations of the spheres.

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