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Patterns of Sustained Oscillations in Neural Networks with Delayed Interactions

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ABSTRACT

We study the Cohen–Grossberg–Hopfield model of neural networks with delayed interactions when the interconnection matrix has only real and purely imaginary eigenvalues. Two indices, the symmetry index and the antisymmetry index, are introduced and are used to describe the pattern of sustained oscillations caused by the delay. It is shown that the parameter plane of these indices is divided into two regions by a smooth curve across which the patterns of oscillations switch. It is also shown that the stability of sustained oscillations is completely determined by the third-order term of the input-output relation.

1. INTRODUCTION AND THE MODEL

Consider a neural network of n neurons governed by the Cohen–Grossberg–Hopfield equation

$$C_i \frac{d}{ds} u_i(s) = -\frac{1}{R_i} u_i(s) + \sum_{j=1}^{n} T_{ij} f_j(u_j(s)), \qquad 1 \leqslant i \leqslant n, \quad (1.1)$$

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where $u_i(s)$ represents the voltage on the input of the ith neuron, C_i is the input capacitance of the cell memberance of the ith neuron, T_{ij} is the synapse efficacy between neuron i and j with $T_{ij} = R_{ij}^{-1}$ when the noninverting output of neuron j is connected to the input of neuron i through a resistance R_{ij} and $T_{ij} = -R_{ij}^{-1}$ when the inverting output of neuron j is connected to the input of neuron i through a resistance R_{ij}, R_i is the total parallel transmemberance resistance of the ith neuron, and

$$\frac{1}{R_i} = \frac{1}{\rho_i} + \sum_{j=1}^{n} \frac{1}{R_{ij}}, \qquad 1 \leqslant i \leqslant n, \tag{1.2}$$

with ρ_i denoting the input resistance of neuron i corresponding to the connection to the outside of the network, and the transfer function $f_i(u)$ is called the input-output relation which is nonlinear and sigmoidal, saturating at ± 1 with the maximal slope at $u = 0$. Throughout this paper, we will assume

$$f_i\colon R \to R \text{ is } C^4, \qquad f_i(0) = 0, \qquad f_i'(0) = \sup_{x \in R} f_i'(x) > 0, \qquad f_i''(0) = 0. \tag{1.3}$$

It was shown in [1, 2], by using a certain Lyapunov function incorporated with the LaSalle's invariance principle, that every solution of system (1.1) with symmetric interconnection matrix (T_{ij}) and without self connections (i.e., $T_{ii} = 0$ for $1 \leqslant i \leqslant n$) converges to the set of equilibria. This leads Hopfield to the conclusion that "neurons with graded response have collective computational properties."

However, it was implicitly observed in [2] and explicitly pointed out by Marcus and Westervelt in [3, 4] that in both real neural networks and their hardware implementations, neurons do not respond and communicate instantaneously and time lags always exist. This motivated the following model of neural networks with delayed interactions:

$$C_i \frac{d}{ds} u_i(s) = -\frac{1}{R_i} u_i(s) + \sum_{j=1}^{n} T_{ij} f_j\big(u_j(s - \tau_{ij}^*)\big), \qquad 1 \leqslant i \leqslant n, \tag{1.4}$$

which has been investigated in [3–10]. In particular, Marcus and Westervelt demonstrated that the delay may cause sustained oscillations in the network. This demonstration was based on linear analysis, numerical integration, and experiments on a small (eight) electric network, and was later rigorously confirmed in [10] (at least for some specific networks) where large-amplitude sustained oscillations in the form of discrete waves were established.

As stable solutions of (1.1) or (1.4) are related to stored information, it is important to describe the pattern and stability of the aforementioned delay induced sustained oscillations. This problem was studied in [5, 8] for networks consisting of a single neuron with self connection. However, the total number of neurons in real networks is immense, and hence the study of Belair and Herz clearly has to be extended from scalar equations to systems.

In this paper we will describe the pattern and stability of sustained oscillations for a general class of networks whose interconnection matrices have only real and purely imaginary eigenvalues. This class of networks includes not only those with symmetric interconnection matrices and antisymmetric interconnection matrices, but also others as well. This restriction on the connection topology allows us to introduce two important quantities, the symmetry index and the antisymmetry index, in terms of the coefficients of the interconnection matrix. It will be shown that the pattern of oscillations is completely characterized by these two indices, the neuron gain and the size of the delay, and that the stability of oscillations is determined by the third-order term of the input-output relation. Moreover, we will also show that in the parameter plane of the symmetry and antisymmetry indices, there is a smooth curve across which patterns of oscillations may switch (this phenomenon is called mode jumping in the monograph [11]).

For the sake of simplicity, we will assume throughout the remainder of this paper that $C_i = C$, $f_i = f$, $\tau^*_{ij} = \tau^*$, and $R_i = R$ for all $1 \leqslant i, j \leqslant n$. Rescaling the time, delay, and T_{ij} by

$$t = s/RC, \quad \tau = \tau^*/RC, \quad J_{ij} = RT_{ij}, \tag{1.5}$$

(1.4) becomes

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{j=1}^{n} J_{ij} f\big(u_j(t-\tau)\big), \qquad 1 \leqslant i \leqslant n. \tag{1.6}$$

Consequently, it is the relative size of the delay (relative to RC, the relaxation time) not the absolute size of the delay that plays a role in the occurrence of sustained oscillations. As designing an electric network to operate more quickly increases the relative size of the intrinsic delay τ, large delay may indeed occur in hardware implementations and hence may cause oscillations. For the discussion of delay phenomenon in real neural networks, we refer to [12].

Note also that in (1.6) we have

$$\sum_{j=1}^{n} |J_{ij}| = R_i \sum_{j=1}^{n} |T_{ij}| = R_i \sum_{j=1}^{n} \frac{1}{R_{ij}}$$

$$= R_i\left(\frac{1}{R_i} - \frac{1}{\rho_i}\right) = 1 - \frac{R}{\rho}. \tag{1.7}$$

Furthermore, (1.6) can be put in the vector form

$$\dot{u}(t) = -u(t) + JF(u(t-\tau)), \tag{1.8}$$

where $u = (u_1, \dots, u_n)^T$, $J = (J_{ij})$ and $F(u) = (f(u_1), \dots, f(u_n))^T$.

2. CRITICAL VALUES OF DELAY

Clearly, 0 is an equilibrium of system (1.8). In order to investigate periodic solutions bifurcating from this trivial solution, we linearize system (1.8) at 0 and obtain

$$\dot{u}(t) = -u(t) + \beta Ju(t-\tau), \tag{2.1}$$

where

$$\beta = f'(0) > 0$$

is the neuron gain. The characteristic equation of (2.1) is given by

$$\det\left[(z+1)\mathrm{Id} - \beta e^{-z\tau} J\right] = 0, \tag{2.2}$$

where Id is the $n \times n$ identity matrix. It is easy to show that z is a solution of (2.2) (i.e., a characteristic value of (2.1)) if and only if there exists $\lambda \in \sigma(J)$, the set of eigenvalues of J, such that

$$z + 1 - \lambda\beta e^{-z\tau} = 0. \tag{2.3}$$

Therefore, if λ_j, $1 \leqslant j \leqslant n$, are eigenvalues of J, then (2.2) is equivalent to n scalar equations

$$z + 1 - \lambda_j \beta e^{-z\tau} = 0, \qquad 1 \leqslant j \leqslant n. \tag{2.4}$$

The following result was established in [3–5].

LEMMA 2.1. *Let $\beta = f'(0)$ and λ_j, $1 \leqslant j \leqslant n$, be eigenvalues of J. Then*

(i) *All roots of* (2.2) *have negative real parts for $\tau = 0$ if and only if* $\operatorname{Re} \lambda_j < 1/\beta$, $1 \leqslant j \leqslant n$;

(ii) *All roots of* (2.2) *have negative real parts for all nonnegative τ if and only if $|\lambda_j| < 1/\beta$, $1 \leqslant j \leqslant n$.*

In what follows, we will regard the delay as the parameter of bifurcation and we assume

$$\max_{1 \leqslant j \leqslant n} \operatorname{Re} \lambda_j < \frac{1}{\beta} < \max_{1 \leqslant j \leqslant n} |\lambda_j|. \tag{2.5}$$

By Lemma 2.1, this assumption says that the trivial solution of (1.8) is asymptotically stable if no delay is present, and this asymptotically stable state can be destabilized by the introduction of time lags.

In what follows, we will restrict ourselves to the network whose (normalized) interconnection matrix J has only real and purely imaginary eigenvalues. Prototypes of such networks include those with symmetric or antisymmetric interconnection matrices. But examples in later sections should show that there are many other types of networks which have only real and purely imaginary eigenvalues.

Since J is a real matrix, its imaginary eigenvalues must appear in pairs. So, we may assume

$$\sigma(J) = \{\alpha_1, \ldots, \alpha_m, \pm i\delta_1, \ldots, \pm i\delta_r\}$$

with

$$\alpha_1 \leqslant \alpha_2 \leqslant \cdots \alpha_s < 0 \leqslant \alpha_{s+1} \leqslant \ldots, \leqslant \alpha_m, \tag{2.6}$$

and

$$0 \leqslant \delta_r \leqslant \delta_{r-1} \leqslant \cdots \leqslant \delta_2 \leqslant \delta_1. \tag{2.7}$$

By (2.5), we must have

$$\alpha_m < \frac{1}{\beta} \tag{2.8}$$

and

$$\alpha_1 < -\frac{1}{\beta}, \tag{2.9}$$

or

$$\delta_1 > \frac{1}{\beta}. \tag{2.10}$$

In what follows, α_1 and δ_1 will be called *the symmetry index* and *the antisymmetry index* of the network (1.8), respectively. It will be shown that the critical value of the delay where the trivial solution is destabilized is completely described in terms of these indices and the neuron gain.

LEMMA 2.2. *Assume that* (2.9) *holds. For each* α_k *with* $\alpha_k < -1/\beta$, *define*

$$\tau(\alpha_k) = \frac{1}{\sqrt{\beta^2\alpha_k^2 - 1}} \arccos \frac{1}{\beta\alpha_k}. \tag{2.11}$$

Then

(i) *At* $\tau = \tau(\alpha_k)$, (2.3) *with* $\lambda = \alpha_k$ *has a pair of purely imaginary simple roots* $\pm i\sqrt{\beta^2\alpha_k^2 - 1}$ *and all other roots have negative real parts*;

(ii) *For each* $\tau \in [0, \tau(\alpha_k))$, *all roots of* (2.3) *with* $\lambda = \alpha_k$ *have negative real parts*;

(iii) *There exists a smooth curve* $z = z(\tau)$ *of zeros of* (2.3) *with* $\lambda = \alpha_k$ *in a neighborhood of* $(\tau(\alpha_k), i\sqrt{\beta^2\alpha_k^2 - 1})$ *such that* $z(\tau(\alpha_k)) = i\sqrt{\beta^2\alpha_k^2 - 1}$ *and* $\mathrm{Re}\, d/d\tau\, z(\tau)|_{\tau=\tau(\alpha_k)} > 0$;

(iv) $\tau(\alpha_k) > \tau(\alpha_1)$ *if* $\alpha_1 < \alpha_k < -1/\beta$.

PROOF. Substituting $z = i\omega$, $\omega > 0$ into (2.3) with $\lambda = \alpha_k$ leads to

$$\begin{aligned} 1 &= \beta\alpha_k \cos(\omega\tau), \\ -\omega &= \beta\alpha_k \sin(\omega\tau) \end{aligned} \tag{2.12}$$

from which we can easily obtain

$$\omega = \sqrt{\beta^2\alpha_k^2 - 1},$$

$$\tau = \frac{1}{\sqrt{\beta^2\alpha_k^2 - 1}}\left[\arccos\frac{1}{\beta\alpha_k} + 2n\pi\right],$$

where $0 < \arccos 1/(\beta\alpha_k) < \pi$ and n is an integer. The least such positive τ is given by

$$\tau = \tau(\alpha_k) = \frac{1}{\sqrt{\beta^2\alpha_k^2 - 1}}\arccos\frac{1}{\beta_k}.$$

This shows that at $\tau(\alpha_k)$, (2.3) with $\lambda = \alpha_k$ has a pair of purely imaginary zeros. Let $H_{\alpha_k}(z,\tau) = z + 1 - \beta\alpha_k e^{-z\tau}$. Then $(\partial H_{\alpha_k})/\partial z = 0$ takes the form $1 + \beta\alpha_k\tau e^{-z\tau} = 0$. Consequently, $H_{\alpha_k} = 0$ and $(\partial H_{\alpha_k})/\partial z = 0$ imply $\tau(z+1) = -1$, i.e., $\tau = -1/(z+1)$. This shows that any multiple zero of $H_{\alpha_k}(z,\tau) = 0$ must be real, and hence $i\sqrt{\beta^2\alpha_k^2 - 1}$ is a simple root of (2.3) with $\alpha = \alpha_k$.

We now show that (2.3) with $\lambda = \alpha_k$ has no root with positive real part. Suppose, for the sake of contradiction, that $z = x + iy$ is a solution of (2.3) with $\lambda = \alpha_k$ such that $x > 0$. As roots of (2.3) continuously depend on τ, using the argument in Lemma 2.1 of [13], we can show that there must exist $\hat{\tau} \in (0, \tau(a_k))$ such that (2.3) with $\lambda = \alpha_k$ and $\tau = \hat{\tau}$ has a purely imaginary root (note that zero is not a root of (2.3) with $\alpha = \alpha_k$ and $\tau = \tau(\alpha_k)$). This contradicts the choice of $\tau(\alpha_k)$, and thus proves (i). A similar argument can be employed to justify (ii).

As $i\sqrt{\beta^2\alpha_k^2 - 1}$ is a simple root of (2.3) with $\lambda = \alpha_k$ and $\tau = \tau(\alpha_k)$ and as $(\partial H_{\alpha_k}(z,\tau))/\partial z \neq 0$ at $z = i\sqrt{\beta^2\alpha_k^2 - 1}$ and $\tau = \tau(\alpha_k)$, the existence of the smooth curve $z = z(\tau)$ of roots of (2.3) is guaranteed. Substituting this value into (2.3) and then differentiating both sides lead to

$$\left.\frac{dz(\tau)}{\partial\gamma}\right|_{\tau=\tau(\alpha_k)} = -\left.\frac{\beta\alpha_k z(\tau)e^{-z(\tau)\tau}}{1 + \beta\alpha_k\tau e^{-z(\tau)\tau}}\right|_{\tau=\tau(\alpha_k)}.$$

Therefore, simple calculation yields

$$\mathrm{Re}\frac{dz(\tau)}{d\tau}\Big|_{\tau=\tau(\alpha_k)} = \frac{\beta^2\alpha_k^2 - 1}{[1 + \tau(\alpha_k)]^2 + \tau^2(\alpha_k)(\beta^2\alpha_k^2 - 1)} > 0.$$

This justifies (iii).

The functions $h_1(x) = 1/(\sqrt{x^2 - 1})$ and $g_1(x) = \arccos 1/x$ are clearly positive and increasing in $x \in (-\infty, -1)$, and so is their composition. This, together with the definition of $\tau(\alpha_k)$, guarantees the monotonicity that $\tau(\alpha_k) > \tau(\alpha_1)$ if $\alpha_1 < \alpha_k < -1/\beta$. This completes the proof.

Similarly, we can establish the following:

LEMMA 2.3. *Assume that* (2.10) *is satisfied. For each* δ_j *with* $\delta_j > 1/\beta$, *define*

$$\tau(\delta_j) = \frac{1}{\sqrt{\beta^2\delta_j^2 - 1}} \arccos \frac{1}{\beta\delta_j}.$$

Then

(i) *At* $\tau = \tau(\delta_j)$, (2.3) *with* $\lambda = i\delta_j$ *has a pair of purely imaginary simple roots* $\pm i\sqrt{\beta^2\delta_j^2 - 1}$ *and all other roots have negative real parts*;

(ii) *For each* $\tau \in [0, \tau(\delta_j))$, *all roots of* (2.3) *with* $\lambda = i\delta_j$ *have negative real parts*;

(iii) *There exists a smooth curve* $z = z(\tau)$ *of zeros of* (2.3) *with* $\lambda = \delta_j$ *in a neighborhood of* $(\tau(\delta_j), i\sqrt{\beta^2\delta_j^2 - 1})$ *such that* $z(\tau(\delta_j)) = i\sqrt{\beta^2\delta_j^2 - 1}$ *and* Re $d/d\tau\, z(\tau)|_{\tau = \tau(\delta_j)} > 0$;

(iv) $\tau(\delta_j) > \tau(\delta_1)$ *if* $\delta_1 > \delta_j > 1/\beta$.

By using Lemma 2.2 and Lemma 2.3, we now know that the lease value of the delay at which the trivial solution is destabilized is given by

$$\tau^* = \min\{\tau(\alpha_1), \tau(\delta_1)\} \tag{2.13}$$

if both (2.9) and (2.10) are satisfied. Let $x^* = x^*(\alpha_1)$ be the unique solution in $(0, \pi/2)$ of the equation

$$x \tan x = \frac{1}{\sqrt{\beta^2\alpha_1^2 - 1}} \arccos \frac{1}{\beta\alpha_1}. \tag{2.14}$$

Then we have

LEMMA 2.4.

$$\tau^* = \begin{cases} \tau(\delta_1) & \text{if } \delta_1 > \dfrac{1}{\beta \sin x^*(\alpha_1)} \\ \tau(\alpha_1) & \text{if } \dfrac{1}{\beta} < \delta_1 \leqslant \dfrac{1}{\beta \sin x^*(\alpha_1)}. \end{cases} \tag{2.15}$$

PROOF. The proof is straightforward after we rewrite $\tau(\delta_1) = z \tan z$ for $z = \arcsin 1/(\beta\alpha_1)$.

Note that the smooth curve Γ given by

$$\delta_1 = \frac{1}{\beta \sin x^*(\alpha_1)}, \qquad \alpha_1 < -\frac{1}{\beta} \tag{2.16}$$

is decreasing and divides the meaningful region $\{(\alpha_1, \delta_1);\ \alpha_1 < -1/\beta,\ \delta_1 > 1/\beta\}$ in the parameter (α_1, δ_1)-plane into two parts in the upper part of which $\tau^* = \tau(\delta_1)$ and in the lower part of which $\tau^* = \tau(\alpha_1)$. As will be shown in next section, this indicates the switch of mode of oscillations.

Note also that generically we should have

$$\delta_1 \neq \frac{1}{\beta \sin x^*(\alpha_1)} \qquad \text{for } \alpha_1 < -\frac{1}{\beta}. \tag{2.17}$$

Under this condition, we can use Lemma 2.2 and Lemma 2.3 and the general Hopf bifurcation theory for functional differential equations (see, cf. [14, 15]) to obtain the following:

THEOREM 2.1. *If* (2.9), (2.10), *and* (2.17) *are satisfied, then system* (1.8) *has a branch of periodic solutions bifurcating from the trivial solution near* $\tau = \tau^*$.

The bifurcation direction and the pattern of oscillation and its stability will be addressed in the next section.

3. DIRECTION, PATTERN, AND STABILITY OF BIFURCATIONS

Rescaling the time again by $v(t) = u(\tau t)$ in (1.8), we get

$$\dot{v}(t) = \tau[-v(t) + JF(v(t-1))]. \tag{3.1}$$

Let $\tau = \tau^* + \mu$. Then we get

$$\dot{v}(t) = (\tau^* + \mu)[-v(t) + JF(v(t-1))], \tag{3.2}$$

where μ is the bifurcation parameter and Theorem 2.1 ensures the existence of a Hopf bifurcation from the trivial solution near $\mu = 0$. The main focus in this section is to establish the algorithm to determine the direction, pattern, and stability of the Hopf bifurcation.

We first consider the case where $\tau^* = \tau(\alpha_1) < \tau(\delta_1)$. In this case, $\tau^* = \tau(\alpha_1) = (1/\sqrt{\beta^2\alpha_1^2 - 1})\arccos 1/(\beta\alpha_1)$ and at $\mu = 0$, the linearization of (3.2) at the trivial solution has a pair of purely imaginary characteristic values $\pm i\omega_0 = \pm i\tau^*\sqrt{\beta^2\alpha_1^2 - 1} = \pm i \arccos 1/(\beta\alpha_1)$. Clearly, there exists a nonsingular matrix P such that

$$J_\alpha := P^{-1}JP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & M \end{pmatrix}. \tag{3.3}$$

Then the transformation $v(t) = Py(t)$ leads to

$$\dot{y}(t) = (\tau^* + \mu)[-y(t) + J_\alpha P^{-1}F(Py(t-1))]. \tag{3.4}$$

Let

$$\eta(\theta, \mu) = (\tau^* + \mu)[-\delta(\theta)\mathrm{Id} + \beta\delta(\theta + 1)J_\alpha], \qquad -1 \leqslant \theta \leqslant 0,$$

where δ is the Dirac function. For $\phi \in C([-1, 0]; R^n)$, define

$$h(\mu, \phi) = (\tau^* + \mu)[J_\alpha P^{-1}F(P\phi(-1)) - \beta J_\alpha \phi(-1)],$$

and

$$R(\mu)\phi = \begin{cases} 0 & -1 \leqslant \theta < 0, \\ h(\mu, \phi) & \theta = 0. \end{cases}$$

Let

$$\mathrm{Dom}(A(\mu)) = \left\{\phi \in C^1([-1,0];\, R^n);\, \dot{\phi}(0^-) = \int_{-1}^{0} d\eta(s,\mu)\phi(s)\right\},$$

$$A(\mu)\phi = \dot{\phi}.$$

Then (3.4) can be formally rewritten as

$$\dot{y}_t = A(\mu)\, y_t + R(\mu)\, y_t. \tag{3.5}$$

For $\psi \in C([0,1];\, C^n)$ and $\phi \in C([-1,0];\, C^n)$, define an inner product by

$$\langle \psi, \phi\rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=-1}^{0}\int_{s=0}^{\theta} \bar{\psi}^T(s-\theta)\, d\eta(\theta,0)\phi(s)\, ds.$$

Let $A^*(0)$ be the adjoint operator of $A(0)$ relative to the above inner product. Then

$$\mathrm{Dom}(A^*(0)) = \left\{\psi \in C^1([0,1];\, C^n);\, \dot{\psi}(0^+) = -\int_{-1}^{0} d\eta^T(s,0)\psi(-s)\right\},$$

$$A^*(0)\psi = -\dot{\psi}(s), \qquad 0 \leqslant s \leqslant 1, \qquad \psi \in \mathrm{Dom}(A^*(0)).$$

It is straightforward to show that $i\omega_0$ is an eigenvalue of $A(0)$ and $A^*(0)$ with corresponding eigenvectors given by $q(\theta) = e^{i\omega_0\theta}(1,0,\dots,0)^T$, $-1 \leqslant \theta \leqslant 0$, and by $q^*(s) = e^{i\omega_0 s}(1+\tau^* - i\omega_0)^{-1}(1; 0,\dots,0)$, $0 \leqslant s \leqslant 1$. Moreover, $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$.

In what follows, we follow the notations and algorithm developed in [15] for general functional differential equations.

Let

$$z(t) = \langle q^*, y_t\rangle,$$

$$w(z,\bar{z})(\theta) = y_t(\theta) - 2\,\mathrm{Re}(zq(\theta)).$$

Then at $\mu = 0$, (3.5) is reduced to an ordinary differential equation for a single complex variable

$$\dot{z}(t) = i\omega_0\, z(t) + \bar{q}^*(0)\hat{h}(z,\bar{z}), \tag{3.6}$$

where

$$\hat{h}(z, \bar{z}) = h(w(z, \bar{z}) + 2\,\mathrm{Re}(zq)).$$

Rewrite (3.6) as

$$\dot{z}(t) = i\omega_0 z(t) + g(z, \bar{z}) \tag{3.7}$$

with

$$w(z, \bar{z}) = w_{20}\frac{z^2}{2} + w_{11} z\bar{z} + w_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{3.8}$$

and

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.9}$$

It is crucial to calculate explicitly g_{21} which we carry out in the following. First of all, note that $f''(0) = 0$ and hence

$$\begin{aligned} h(0, y_t) &= \tau^*\left[J_\alpha P^{-1}(Py_t(-1)) - \beta J_\alpha y_t(-1)\right] \\ &= \tau^* J_\alpha P^{-1} \frac{1}{6} f'''(0) \begin{pmatrix} \left(\sum_{j=1}^{n} p_{1j} y_j(t-1)\right)^3 \\ \vdots \\ \left(\sum_{j=1}^{n} p_{nj} y_j(t-1)\right)^3 \end{pmatrix} + \text{h.o.t.} \end{aligned} \tag{3.10}$$

Moreover, we know that

$$\begin{aligned} y_t(\theta) &= w(z, \bar{z})(\theta) + zq(\theta) + \bar{z}\bar{q}(\theta) \\ &= w(z, \bar{z})(\theta) + ze^{i\omega_0\theta}(1,0,\dots,0)^T + \bar{z}e^{-i\omega_0\theta}(1,0,\dots,0)^T. \end{aligned} \tag{3.11}$$

Let

$$b = \frac{1}{1 + \tau^* - i\omega_0} \tag{3.12}$$

and

$$C = (C_1, \ldots, C_n)^T = (c_{ij}) = P^{-1}.$$

Then

$$\begin{aligned}
g(z, \bar{z}) &= \bar{q}^*(0)\hat{h}(z, \bar{z}) \\
&= \bar{q}^*(0)h(0, y_t) \\
&= \frac{\bar{b}\tau^* f'''(0)}{6}\alpha_1 C_1 \begin{pmatrix} \left(\sum_{j=1}^n p_{1j} y_j(t-1)\right)^3 \\ \vdots \\ \left(\sum_{j=1}^n p_{nj} y_j(t-1)\right)^3 \end{pmatrix} + \text{h.o.t.} \qquad (3.13) \\
&= \frac{\bar{b}\tau^* f'''(0)\alpha_1}{6} \sum_{k=1}^n c_{1k}\left(\sum_{j=1}^n p_{kj} y_j(t-1)\right)^3 + \text{h.o.t.} \\
&= \frac{\bar{\tau}^* f'''(0)\alpha_1}{6} \sum_{k=1}^n c_{1k} p_{k1}^3 (ze^{-i\omega_0} + \bar{z}e^{i\omega_0})^3 + \text{h.o.t..}
\end{aligned}$$

Therefore, we have obtained

LEMMA 3.1.

$$\begin{aligned}
g_{21} &= 2\frac{\bar{b}\tau^* f'''(0)\alpha_1}{6} \sum_{k=1}^n c_{1k} p_{k1}^3 3 e^{-i\omega_0} \\
&= \bar{b}\tau^* f'''(0)\alpha_1\left(\sum_{k=1}^n c_{1k} p_{k1}^3\right) e^{-i\omega_0}.
\end{aligned}$$

Also, by direct calculation, we can establish

LEMMA 3.2. *If $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ is the smooth curve of zeros of $H(\lambda, \mu) = (\lambda + \tau^* + \mu) - \beta(\tau^* + \mu)\alpha_1 e^{-\lambda}$ such that $\alpha(0) = 0$, $\omega(0) =$*

ω_0, *then*

$$\alpha'(0) = \frac{\omega_0^2}{\tau^*\left[(1+\tau^*)^2 + \omega_0^2\right]} > 0,$$

$$\omega'(0) = \frac{\omega_0(1+\tau^*)}{\tau^*\left[(1+\tau^*)^2 + \omega_0^2\right]} > 0.$$

According to the general theory in [15], the bifurcation direction and stability are determined by

$$\mu_2 = -\frac{\operatorname{Re} c_1(0)}{\alpha'(0)} = -\frac{\operatorname{Re} g_{21}}{2\alpha'(0)},$$

and

$$\beta_2 = 2\operatorname{Re} c_1(0) = \operatorname{Re} g_{21}.$$

Therefore, we have

THEOREM 3.1. *Assume that $\tau(\alpha_1) < \tau(\delta_1)$ and*

$$\sigma_\alpha := \operatorname{Re}\left[\bar{b}\left(\sum_{k=1}^{n} c_{1k} p_{k1}^3\right) e^{-i\omega_0}\right] f'''(0) \neq 0. \tag{3.14}$$

Then system (1.8) *has a Hopf bifurcation of periodic solutions at $\tau^* = \tau(\alpha_1)$. This bifurcation is supercritical* (*resp., subcritical*) *and the periodic solutions are asymptotically stable* (*resp., unstable*) *if $\sigma_\alpha < 0$* (*resp., $\sigma_\alpha > 0$*). *Moreover, the periodic solutions have the following representation:*

$$\mu(t,\tau) = 2\sqrt{\frac{2\alpha'(0)}{-\sigma_\alpha}(\tau - \tau^*)}\; P\operatorname{Re}(1,0,\ldots,0)^T e^{i\sqrt{\beta^2\alpha_1^2 - 1}}$$
$$+ O(|\tau - \tau^*|). \tag{3.15}$$

In the case where $\tau(\delta_1) < \tau(\alpha_1)$, we can find a nonsingular matrix Q such that

$$J_\delta := Q^{-1}JQ = \begin{pmatrix} 0 & \delta_1 & 0 \\ -\delta_1 & 0 & 0 \\ 0 & 0 & N \end{pmatrix}.$$

Using similar arguments, we can show that

$$g_{21} = D\tau^* f'''(0)\,\delta_1 e^{-i\omega_0}\delta_1^4 \sum_{k=1}^{n} (ia_{2k} - a_{1k})(q_{k1} + iq_{k2})^2 (q_{k1} - iq_{k2}),$$

where

$$D = \frac{(1+\tau^*)i - \omega_0}{2\delta_1^2\left[(1+\tau^*)^2 + \omega_0^2\right]},$$

$$Q = (q_{ij}), \qquad Q^{-1} = (a_{ij}).$$

Therefore, we have

THEOREM 3.2. *Assume that* $\tau(\alpha_1) > \tau(\delta_1)$ *and*

$$\sigma_\delta := \mathrm{Re}\left[D\tau^* e^{-i\omega_0} \sum_{k=1}^{n} (ia_{2k} - a_{1k})(q_{k1} + iq_{k2})^2 (q_{k1} - iq_{k2})\right] f'''(0) \neq 0. \tag{3.16}$$

Then system (1.8) *has a Hopf bifurcation of periodic solutions at* $\tau^* = \tau(\delta_1)$. *This bifurcation is supercritical* (*resp., subcritical*) *and the periodic solutions are asymptotically stable* (*resp., unstable*) *if* $\sigma_\delta < 0$ (*resp.,* $\sigma_\delta > 0$). *Moreover, the periodic solutions have the following representation:*

$$\mu(t,\tau) = 2\sqrt{\frac{2\alpha'(0)}{-\sigma_\delta}(\tau - \tau^*)}\; Q\,\mathrm{Re}(i\delta_1, \delta_1, 0, \dots, 0)^T e^{i\sqrt{\beta^2\delta_1^2 - 1}}$$
$$+\, O(|\tau - \tau^*|). \tag{3.17}$$

Summarizing Theorem 3.1 and Theorem 3.2, we can conclude that (i) the bifurcation direction and stability are determined by the third order term $f'''(0)$ of the input-output relation; (ii) as (α_1, δ_1) moves across the smooth curve $\delta_1 = 1/(\beta \sin x^*(\alpha_1))$ defined in the last section, the mode of bifurcated periodic solutions switch from (3.15) to (3.17).

4. SOME EXAMPLES

We demonstrate the general results (Theorem 3.1 and Theorem 3.2) by two examples.

EXAMPLE 4.1. We consider the all-inhibitory network, that is, system (1.8) with

$$J = \frac{1}{n-1}\begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & 0 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & 0 \end{pmatrix}, \qquad n \geqslant 3.$$

In this case,

$$\sigma(J) = \left\{-1, \frac{1}{n-1}, \dots, \frac{1}{n-1}\right\}$$

and hence (2.8) and (2.9) hold if

$$\frac{1}{n-1} < \frac{1}{\beta} < 1. \tag{4.1}$$

Therefore, the critical value of the delay is given by

$$\tau^* = \tau(\alpha_1) = \frac{1}{\sqrt{\beta^2 - 1}} \arccos\left(-\frac{1}{\beta}\right)$$

and the associated purely imaginary characteristic values are

$$\pm i\omega_0 = \pm i \arccos\left(-\frac{1}{\beta}\right).$$

Note

$$\bar{b} = \frac{1}{1 + \tau^* + i\omega_0} = \frac{(1+\tau^*) - i\omega_0}{(1+\tau^*)^2 + \omega_0^2}$$

and

$$c_{1k} = p_{k1} = \frac{1}{\sqrt{n}}$$

as J is symmetric and $v = (1, 1, \dots, 1)^T$ is an eigenvector of J. Therefore,

$$\begin{aligned} g_{21} &= -\bar{b}\tau^* f'''(0)\left(\sum_{k=1}^{n} c_{1k} p_{k1}^3\right) e^{-i\omega_0} \\ &= -\tau^* f'''(0) \frac{(1+\tau^*) - i\omega_0}{n\left[(1+\tau^*)^2 + \omega_0^2\right]} e^{-i\omega_0}. \end{aligned}$$

This implies that

$$\begin{aligned} \sigma_\alpha &= \operatorname{Re} g_{21} \\ &= -\frac{\tau^* f'''(0)}{n\left[(1+\tau^*)^2 + \omega_0^2\right]} \left[(1+\tau^*)\cos\omega_0 - \omega_0 \sin\omega_0\right] \\ &= -\frac{\tau^* f'''(0)}{n\left[(1+\tau^*)^2 + \omega_0^2\right]} \left[(1+\tau^*)\left(-\frac{1}{\beta}\right) - \omega_0 \sin\omega_0\right] \\ &= -\frac{\tau^*\left[(1+\tau^*)\dfrac{1}{\beta} + \omega_0 \sin\omega_0\right]}{n\left[(1+\tau^*)^2 + \omega_0^2\right]} f'''(0). \end{aligned}$$

Consequently, by Theorem 3.1, if $f'''(0) < 0$ then system (1.8) has a supercritical Hopf bifurcation of asymptotically stable periodic solutions.

EXAMPLE 4.2. We now consider system (1.8) with

$$J = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Note that J is neither symmetric nor antisymmetric, but $\sigma(J) = \{-\frac{1}{2}, \pm i\frac{1}{\sqrt{2}}\}$, and hence the symmetry index $\alpha = -\frac{1}{2}$ and the antisymmetry index $\delta = \frac{1}{\sqrt{2}}$. We assume $\beta = f'(0) > 2$. Then (2.8)–(2.10) are satisfied. Note also that if

$$\frac{1}{\beta \sin x} \geqslant \delta = \frac{1}{\sqrt{2}},$$

the

$$\sin x \leqslant \frac{\sqrt{2}}{\beta} < \frac{1}{\sqrt{2}}$$

and hence $x < \pi/4$, $x \tan x < \pi/4$. On the other hand,

$$\frac{1}{\sqrt{\beta^2\alpha^2 - 1}} \arccos \frac{1}{\beta\alpha} = \left(\frac{\pi}{2} + \theta\right) \tan \theta > \frac{\pi}{4},$$

where $\theta = \arcsin 2/\beta$. Therefore, by Lemma 2.4, we conclude that the critical value of the delay is given by

$$\tau^* = \tau(\delta_1) = \frac{\sqrt{2}}{\sqrt{\beta^2 - 1}} \arcsin \frac{\sqrt{2}}{\beta}$$

and the associated purely imaginary characteristic values are

$$\pm i\omega_0 = \pm i \arcsin \frac{\sqrt{2}}{\beta}.$$

Let

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then

$$A = Q^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$J_\delta = Q^{-1} J Q = \begin{pmatrix} 0 & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\dfrac{1}{2} \end{pmatrix}.$$

Comparing with Theorem 3.2, we have

$$a_{11} = 1, \qquad a_{12} = a_{13} = 0,$$

$$a_{21} = 0, \qquad a_{22} = \frac{1}{\sqrt{2}}, \qquad a_{23} = -\frac{1}{\sqrt{2}},$$

$$q_{11} = 1, \qquad q_{21} = q_{31} = 0,$$

$$q_{12} = 0, \qquad q_{22} = \frac{1}{\sqrt{2}}, \qquad q_{32} = -\frac{1}{\sqrt{2}},$$

$$D = \frac{(1+\tau^*)i - \omega_0}{2(1/\sqrt{2})^2\left[(1+\tau^*)^2 + \omega_0^2\right]}.$$

Therefore,

$$\sigma_\delta = \tfrac{11}{16}\tau^*\left[\,\omega_0 \cos \omega_0 - (1+\tau^*)\sin \omega_0\right] f'''(0).$$

By Theorem 3.2, we can then conclude that if $f'''(0) < 0$ then system (1.8) has a supercritical Hopf bifurcation of asymptotically stable periodic solutions at τ^*.

5. CONCLUSIONS

For the Cohen–Grossberg–Hopfield delayed model of neutral networks whose interconnection matrix has only real and purely imaginary eigenvalues, the symmetry index (the minimal real eigenvalue of the interconnection matrix) and the antisymmetry index (the purely imaginary eigenvalue of the interconnection matrix with maximal norm) are introduced. It is shown that these two indices and the neuron gain determine the critical value of the delay where a Hopf bifurcation occurs and describe the mode switch of the Hopf bifurcation. It is also shown that the third-order term of the input-output relation decide the bifurcation direction and stability of the periodic solutions bifurcating from the trivial solution.

All of the analysis is carried out at the trivial solution and can be applied to any other equilibrium. However, our study is local in nature and concerns only the network dynamics very close to equilibria. The amplitude of periodic solutions obtained is small ($0(\sqrt{\tau - \tau^*})$) and the question whether large-amplitude periodic solutions exist when τ is far away from τ^* is not discussed here. This question was addressed in [9, 10] for networks with very strong symmetry; answering this question for general networks seems to be a challenging problem. Moreover, the structure of the basins of attraction for each periodic solutions obtained is not described and a complete description of the global dynamics is not attempted but should be studied in the future.

A more realistic model should be the one with distributed delay to account the stochastic element in the delayed interaction between neurons, see, cf. [6, 7, 12]. This will be discussed in another paper.

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