DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **17**, Number **7**, October **2012**

doi:10.3934/dcdsb.2012.17.2509

pp. **2509–2522**

ON A REACTION-DIFFUSION MODEL FOR STERILE INSECT RELEASE METHOD WITH RELEASE ON THE BOUNDARY

Xin Li

Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, China

Xingfu Zou

Department of Applied Mathematics, University of Western Ontario
London, Ontario, Canada, N6A 5B7

(Communicated by Yuan Lou)

Abstract. We consider a partial differential equation model that describes the sterile insect release method (SIRM) in a bounded 1-dimensional domain (interval). Unlike everywhere-releasing in the domain as considered in previous works [17] and [14] , we propose the mechanism of releasing on the boundary only. We show existence of the fertile-free steady state and prove its stability under some conditions. By using the upper-lower solution method, we also show that under some other conditions there may exist a coexistence steady state. Biological implications of our mathematical results are that the SIRM with releasing only on the boundary can successfully eradicate the fertile insects as long as the strength of the sterile releasing is reasonably large, while the method may also fail if the releasing is not sufficient.

1. **Introduction.** Among various biological control methods for insects is the Sterile Insect Release Method (SIRM) which was originally suggested by Knipling [15]. The key idea of this method is that the released sterile insects will compete with the fertile individuals for mating, and the competition can reduce the productive capacity of the target species, and may eventually lead to a population crash, eradicating the fertile insects. There have been quite a few successful applications of this method in field conditions against species such as screwworm fly [18, 6, 16], melon fly [12, 13], codling moth [25], and bollworm [10].

To quantitatively assess the effectiveness of SIRM, many mathematical models have been proposed and studied. For example, [1], [2], [4], [5], [7], [8], [11] used models without considering the spatial aspect. On the other hand, realizing the significance of spatial factor in pest control, [17], [19] and [22] proposed partial differential equation models with the diffusion terms accounting for mobility of the insects. Such models have revealed new phenomena that can not be observed in ordinary differential equation models. For example, in [17], the authors explored the

2000 *Mathematics Subject Classification.* Primary: 35J57, 35K57; Secondary: 92B05, 92D25.

Key words and phrases. Sterile insect release method, diffusion, steady state, stability, upper-lower solution, coexistence.

The first author is Supported by China Scholarship Council. The corresponding author is partially supported by NSERC and MITACS of Canada.

combined effects of dispersal terms and growth dynamics, and showed that for realistic parameter values, the PDE model predicts extinction, while the corresponding model ignoring spatial dispersal terms would predict persistence.

The results in [17] on the role of insect dispersal were obtained by investigating existence of traveling wave front solutions to the following PDE model

$$\begin{cases} u_t = d_1 \Delta u + u\left(\dfrac{a_1 u}{u+n} - a_2\right) - 2g\, u(u+n), \\ n_t = d_2 \Delta n + r - a_2 n - 2g\, n(u+n), \end{cases} \qquad t \geq 0, \quad x \in \mathbb{R}. \tag{1.1}$$

Here $u(t,x)$ and $n(t,x)$ denote the densities of fertile and sterile females respectively, a_1 is the birth rate of the fertile insects, a_2 is the density-independent death rate which is smaller than a_1, g is the density-dependent death rate (given by $2g = (a_1 - a_2)/\text{carrying capacity}$). The constants d_1 and d_2 are the diffusion coefficients of the fertile insects and the sterile insects respectively, and r is the constant release rate of the sterile insects. Since the main concern in [22, 17] was traveling wave front solutions to this model, the one dimensional spatial variable x was assumed to be from the whole space $\mathbb{R}$. For details on the biological assumptions under which this model was proposed, see [22, 17].

Two issues arise for the model (1.1): (i) a habitat is bounded in real world; (ii) releasing sterile insects *everywhere* in the habitat by a constant rate r is impractical in reality. Addressing (i), Jiang et al [14] have recently considered the same set of partial differential equations but on a bounded spatial domain with no-flux boundary condition posed on the boundary. In this paper, we address *both* issues (i) and (ii) mentioned above by considering a further alternation of the model (1.1). More precisely, we adopt the two partial differential equations in (1.1) to describe the interaction of the fertile and sterile female insects, but we confine the spatial variable x to a bounded interval $\Omega = (-\ell, \ell)$; moreover, instead of releasing the sterile insects everywhere in Ω, we consider releasing the sterile insects only at the boundary of Ω, i.e., $\partial\Omega = \{-\ell, \ell\}$, with the releasing amount proportional to the gradient of the sterile insects at the two end points. These considerations lead to the following model in the form of Initial-Boundary-Value problem:

$$\begin{cases} u_t = d_1 u_{xx} + u\left(\dfrac{a_1 u}{u+n} - a_2\right) - 2g\, u(u+n), & t > 0,\ x \in \Omega, \\ n_t = d_2 n_{xx} - a_2 n - 2g\, n(u+n), & t > 0,\ x \in \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0,\ \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = r > 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0,\ n(x,0) = n_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where ν is the outward unit direction on the boundary of Ω.

For convenience of analysis, we non-dimensionalize (1.2) by the following

$$\begin{aligned} & x^* = x\sqrt{\frac{a_1}{d_1}},\ t^* = t a_1,\ u^* = u\frac{g}{a_1},\ n^* = n\frac{g}{a_1}, \\ & A = \frac{a_2}{a_1},\ d = \frac{d_2}{d_1},\ R = r\frac{g}{a_1}\sqrt{\frac{d_1}{a_1}},\ L = \ell\sqrt{\frac{a_1}{d_1}}. \end{aligned} \tag{1.3}$$

Dropping asterisks for notational simplicity, (1.2) is transformed to

$$\begin{cases} u_t = u_{xx} + u\left(\dfrac{u}{u+n} - A - 2(u+n)\right), & x \in \Omega, \\ n_t = dn_{xx} - An - 2n(u+n), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \dfrac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = R > 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0,\ n(x,0) = n_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.4}$$

where Ω is now the interval $(-L, L)$.

Note that the Neumann boundary condition for n in (1.4) is not homogeneous due to the new releasing mechanism. Applying the transformation $\hat{n}(x) = n(x) - \frac{R}{2L}x^2$ and still writing $n(x)$ instead of $\hat{n}(x)$ for simplicity, System (1.4) is further transformed to the following system with zero-flux boundary condition:

$$\begin{cases} u_t = u_{xx} + u\left(\dfrac{u}{u+n+\frac{R}{2L}x^2} - A - 2(u+n+\frac{R}{2L}x^2)\right), & x \in \Omega, \\ n_t = dn_{xx} - A(n+\frac{R}{2L}x^2) - 2(n+\frac{R}{2L}x^2)(u+n+\frac{R}{2L}x^2) + \frac{dR}{L}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \dfrac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0,\ n(x,0) = n_0(x) \geq -\frac{R}{2L}x^2, & x \in \Omega. \end{cases} \tag{1.5}$$

In the rest of this paper, we investigate the dynamical behavior of solutions to System (1.5). In Section 2, we address well-posedness of (1.5) including existence, uniqueness and boundedness of solution to (1.5). In Section 3, we first consider the existence and stability of a steady state of the form $(0, n^*(x))$ corresponding to the scenario of eradication of the fertile insects. In Section 4, we explore existence and non-existence of coexistence steady state with the former indicating the failure of the SIRM while the latter implying the success of the SIRM. We conclude the paper by a summary and some discussion in Section 5.

2. **Well-posedness of the model.**

For notational convenience, we denote by $f(x,u,n)$ and $g(x,u,n)$ the two nonlinear functions on the right hand side of (1.5), that is,

$$f(x,u,n) \triangleq u\left(\frac{u}{u+n+\frac{R}{2L}x^2} - A - 2(u+n+\frac{R}{2L}x^2)\right),$$

$$g(x,u,n) \triangleq -A(n+\frac{R}{2L}x^2) - 2(n+\frac{R}{2L}x^2)(u+n+\frac{R}{2L}x^2) + \frac{dR}{L}.$$

In order to consider classic solutions of (1.5), we introduce the space $Y = C^2(\Omega,\mathbb{R}^2) \cap C^1(\bar{\Omega},\mathbb{R}^2)$. According to the biological requirement on the variables u and v, we only need to consider the following subset X in Y:

$$X \triangleq \left\{(u,n) \in Y | u \geq 0,\ n \geq -\frac{R}{2L}x^2, x \in \bar{\Omega}\right\}.$$

The following theorem confirms the well-posedness of (1.5), including existence, uniqueness and boundedness of a solution to (1.5).

Theorem 2.1. *For each $(u_0, n_0) \in X$, there exists a unique solution of system (1.5) and this solution remains in X. This solution is bounded, and hence, exists* *globally (i.e., for all $t \geq 0$). Moreover there hold $u(t,x) > 0$ and $n(t,x) > \frac{-R}{2L}x^2$ for $t > 0$ if $u_0 \not\equiv 0$ and $n_0 \not\equiv \frac{-R}{2L}x^2$.*

Proof. Note that (1.5) is a competitive system and hence, belongs to the quasi-monotone case by the terminologies of [21]. Thus, we can prove this theorem by upper-lower solution method, which is also used for proving a similar theorem in [21] (Theorem 12.4.1) for the Lotka-Volterra competition R-D system.

Let

$$(\underline{u}, \underline{n}) = (0, \frac{-R}{2L}x^2) \quad \text{and} \quad (\bar{u}, \bar{n}) = (M, N)$$

where M and N are constants satisfying

$$M \geq \max\left\{\bar{u}_0, \frac{1-A}{2}\right\}, \ \ \bar{u}_0 = \sup_{x\in\Omega} u_0(x),$$

$$N \geq \max\left\{\bar{n}_0, \frac{1}{4}\left(\sqrt{A^2 + \frac{8dR}{L}} - A\right)\right\}, \ \ \bar{n}_0 = \sup_{x\in\Omega} n_0(x).$$

Then, one can easily verify that

$$\bar{u}_t - \bar{u}_{xx} - f(x, \bar{u}, \underline{n}) = M[2M - (1-A)] \geq 0;$$

$$\bar{n}_t - d\bar{n}_{xx} - g(x, \underline{u}, \bar{n}) = A\left(N + \frac{R}{2L}x^2\right) + 2\left(N + \frac{R}{2L}x^2\right)^2 - \frac{dR}{L} \geq 0;$$

$$\underline{u}_t - \underline{u}_{xx} - f(x, \underline{u}, \bar{n}) = 0 \leq 0;$$

$$\underline{n}_t - d\underline{n}_{xx} - g(x, \bar{u}, \underline{n}) = 0 \leq 0.$$

This shows that $(\underline{u}, \underline{n})$ and $(\bar{u}, \bar{n})$ are an ordered pair of lower-upper solutions of (1.5). By [21, p 397, Theorem 8.3.2], (1.5) has a unique solution (u, n) which is between this pair of lower-upper solutions. Thus it is bounded, and hence exists globally. Moreover, when $u_0 \not\equiv 0$ and $n_0 \not\equiv \frac{-R}{2L}x^2$, by the increasing property of iteration sequence starting from the lower solution in the proof of [21, p 397, Theorem 8.3.2], we conclude that $u(t,x) > 0$ and $n(t,x) \geq \frac{-R}{2L}x^2$ for all $t > 0$. This completes the proof. $\square$

Remark 1. From the proof of the above theorem, we can actually see that for any constants M and N satisfying

$$M \geq \frac{1-A}{2}, \ \ N \geq \frac{1}{4}\left(\sqrt{A^2 + \frac{8dR}{L}} - A\right),$$

the subset

$$X(M,N) \triangleq \left\{(u,n) \in Y | 0 \leq u \leq M, \ \ -\frac{R}{2L}x^2 \leq n \leq N, \ \ x \in \bar{\Omega}\right\}.$$

is positively invariant for (1.5).

3. **Boundary steady state: Eradication of fertile insects.** Steady state of the form $(u^*(x), n^*(x)) = (0, n^*(x))$ accounts for the situation when the fertile population is wiped out, and hence is of practical importance. It is obvious that the existence of such a boundary (located on the n-axis on the u–n plane) steady state for (1.5) is equivalent to the existence of a positive solution to the following elliptic problem

$$\begin{cases} dn_{xx} + g(x, 0, n) = 0, & x \in \Omega, \\ n(x) \geq -\frac{R}{2L}x^2, & x \in \Omega, \\ \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

We first consider a more general elliptic boundary value problem for general space dimension, and establish an existence and uniqueness result for such a problem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}$ be a bounded open domain. Assume that $G(x, n) \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ is decreasing with respect to n. If there exists a unique $\tilde{n}(x) \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R})$ such that $G(x, \tilde{n}(x)) = 0$ on $\bar{\Omega}$, then the boundary condition problem*

$$\begin{cases} \Delta n + G(x, n) = 0, & x \in \Omega, \\ \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

has a unique solution $n^*(x) \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R})$. Moreover, $n^*(x)$ satisfies $\inf_{\bar{\Omega}} \tilde{n}(x) \leq n^*(x) \leq \sup_{\bar{\Omega}} \tilde{n}(x)$, $\forall x \in \bar{\Omega}$.

Proof. Let $\underline{n} = \inf_{\bar{\Omega}} \tilde{n}(x)$ and $\overline{n} = \sup_{\bar{\Omega}} \tilde{n}(x)$. By the monotone property of G, it is easily seen that $\underline{n}$ is a lower solution and $\overline{n}$ is an upper solution of (3.2). By the continuity of G on $\bar{\Omega} \times \mathbb{R}$, there exists a positive constant $K = K(|\Omega|, \underline{n}, \overline{n})$ such that for any $(x, n), (y, m) \in \bar{\Omega} \times [\underline{n}, \overline{n}]$, we have

$$|G(x, n) - G(y, m)| \leq K(|n - m| + |x - y|).$$

By Theorem 3.2.2 in [21], we conclude that there exists a solution $n^*(x)$ of (3.2) satisfying $\underline{n} \leq n^*(x) \leq \overline{n}$, $\forall x \in \bar{\Omega}$.

To prove the uniqueness, we assume that there is another $n^+(x) \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R})$ solving (3.2). By the decreasing (in n) property of $G(x, n)$, we can easily see that for sufficiently large constant $C > 0$, $-C$ and C are a pair of ordered lower and upper solutions of (3.2), satisfying $-C \leq n^*, n^+ \leq C$. Again, by Theorem 3.2.2 in [21], there is a minimal solution $n_1(x)$ and a maximal solution $n_2(x)$ to (3.2) satisfying $n_1 \leq n_2$ and

$$n^*(x), n^+(x) \in [n_1(x), n_2(x)] \subset [-C, C] \qquad x \in \Omega.$$

Let $w = n_2 - n_1$. Subtracting the n_1 equation from the n_2 equation yields

$$-\Delta w = G(x, n_2) - G(x, n_1) \leq 0, \quad x \in \Omega.$$

It is obvious that $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$. Thus, by the Maximal Principle and the Hopf Boundary Point Lemma, we conclude that w is a constant in Ω, and hence $0 = -\Delta w = G(x, n_2) - G(x, n_1)$, $x \in \Omega$. This together with the decreasing (in n) property of $G(x, n)$ implies that $n_1(x) \equiv n_2(x)$, and therefore, $n_+(x) \equiv n^*(x)$ on $\bar{\Omega}$, proving the uniqueness. The proof is completed. ∎

Applying this theorem to (3.1) with $\Omega = (-L, L)$ and $G(x, n) = g(x, 0, n)$, we can obtain the existence of a unique solution to (3.1), and hence the existence of a unique boundary steady state for (1.5), with some estimates for upper and lower bounds of this unique steady state.

Theorem 3.2. *System* (1.5) *always has a unique boundary steady state* $E_0(x) = (0, n^*(x))$ *with* $n^*(x)$ *satisfying*

(i) $\frac{1}{4}\left(\sqrt{A^2 + \frac{8dR}{L}} - A\right) - \frac{1}{2}RL < n^*(x) < \frac{1}{4}\left(\sqrt{A^2 + \frac{8dR}{L}} - A\right), \ x \in \Omega;$

(ii) $\inf_{\partial\Omega} n^*(x) = \inf_{\bar{\Omega}} n^*(x)$, *and there exists a unique* $x_1 \in \Omega$ *such that* $n^*(x_1) = \sup_{\bar{\Omega}} n^*(x)$.

Moreover, E_0 *is locally asymptotically stable.*

Proof. We regard $g(x, 0, n)$ as a quadratic function of n with x as a parameter:

$$g(x, 0, n) = -2n^2 - (\frac{2R}{L}x^2 + A)n - \frac{R^2}{2L^2}x^4 - \frac{AR}{2L}x^2 + \frac{dR}{L}. \tag{3.3}$$

Obviously, as a function of n, $g(x, 0, n)$ is symmetric about and attains its maximum at $l(x) = -\frac{R}{2L}x^2 - \frac{A}{4}$. Since $l(x) = -\frac{R}{2L}x^2 - \frac{A}{4} < -\frac{R}{2L}x^2$, $g(x, 0, n)$ is decreasing in n for all $n \geq -\frac{R}{2L}x^2$. Note that $g(x, 0, n) = 0$ has two roots

$$n_\pm(x) = \frac{1}{4}\left(\pm\sqrt{A^2 + \frac{8dR}{L}} - \frac{2R}{L}x^2 - A\right), \tag{3.4}$$

but only $n_+(x) > -\frac{R}{2L}x^2$ on $\bar{\Omega}$. By Theorem 3.1, (3.1) has a unique solution $n^*(x)$ satisfying $\inf_{\bar{\Omega}} n_+(x) \leq n^*(x) \leq \sup_{\bar{\Omega}} n_+(x)$. The inequalities in (i) follow from the fact that $\inf_{\bar{\Omega}} n_+(x) = n_+(L)$ and $\sup_{\bar{\Omega}} n_+(x) = n_+(0)$. The conclusion in (ii) is a direct result of the Maximum Principle and Hopf Boundary Lemma.

The stability of E_0 is determined by the following eigenvalue problem

$$\begin{cases} -\phi_{xx} - f_u\phi - f_n\psi = \lambda\phi, \\ -d\psi_{xx} - g_u\phi - g_n\psi = \lambda\psi, \\ \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.5}$$

where all the partial derivatives in the coefficients are evaluated at $E_0 = (0, n^*(x))$, that is,

$$\begin{aligned} f_u &= -A - 2(n^* + \tfrac{R}{2L}x^2), & f_n &= 0, \\ g_u &= -2(n^* + \tfrac{R}{2L}x^2), & g_n &= -A. \end{aligned}$$

E_0 is locally asymptotically stable if and only if all the eigenvalues of (3.5) have positive real parts. Since the first equation of (3.5) is decoupled from the second one, the eigenvalue set of (3.5) is a subset of that of the following problem

$$\begin{cases} -\phi_{xx} - f_u\phi = \lambda\phi, \\ \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.6}$$

Because $-f_u = A + 2(n^* + \frac{R}{2L}x^2) > 0$ on $\bar{\Omega}$, we know that the problem (3.6) has a unique principal eigenvalue λ_1 which is real and positive (Theorem 2.4 in [9]), and the real parts of all other eigenvalues are larger than λ_1. Hence, all the real parts of eigenvalues of (3.5) are positive, which implies that E_0 is locally asymptotically stable, completing the proof. $\square$

4. **Existence or non-existence of coexistence steady state: Failure or success of SIRM.** A coexistence steady state of (1.5) is a positive solution to the following elliptic system

$$\begin{cases} u_{xx} + f(x,u,n) = 0, & x \in \Omega, \\ dn_{xx} + g(x,u,n) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \dfrac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $f(x,u,n)$ and $g(x,u,n)$ are given in Section 2. For convenience of notations, we denote

$$\begin{aligned} \tilde{R} &= \frac{L}{216\,d}\left(\sqrt{\tilde{A}^2 + 432A^2(1-A)^2} + \tilde{A}\right), \\ \tilde{A} &= 4(1+A)[2(1-A)^2 - A]. \end{aligned} \tag{4.2}$$

The main result of this section is the following theorem on the existence of a coexistence steady state.

Theorem 4.1. *For given parameters d, A, L, there exists an $R_c = R_c(d,A,L) < \tilde{R}$, such that when $R < R_c$, (4.1) has at least one positive solution.*

Proof. We will prove this theorem by the method of upper-lower solutions. In order to construct the required pair of upper-lower solutions for (4.1), we need some preparations.

Set $z = n + \frac{R}{2L}x^2$ in (4.1), and let $f_0(u,z) = f(0,u,z)$ and $g_0(u,z) = g(0,u,z)$. Solving $g_0(u,z) = 0$ for u in terms of $z > 0$ yields

$$u = \frac{-2z^2 - Az + \frac{dR}{L}}{2z}. \tag{4.3}$$

Similarly, solving $f_0(u,z) = 0$ for u in terms of $z > 0$ leads to either $u = 0$ or

$$u - A(u+z) - 2(u+z)^2 = 0. \tag{4.4}$$

Clearly, (4.4) has two positive real roots $u_\pm = u_\pm(z)$ if and only if $z < z_c \triangleq (1-A)^2/8$, and

$$u_\pm(z) = -z + \frac{1}{4}\left(\pm\sqrt{(1-A)^2 - 8z} + (1-A)\right). \tag{4.5}$$

Moreover, if $z < z_c \triangleq (1-A)^2/8$, then $f(u,z) < 0$ when $u(z) < u_-(z)$ or $u(z) > u_+(z)$, $f(u,z) > 0$ when $u_-(z) < u(z) < u_+(z)$ (see Figure 1).

Substituting (4.3) into (4.4), we can obtain

$$h(z) \triangleq -2z^3 - Az^2 + \delta(1+A)z - \delta^2 = 0, \tag{4.6}$$

where $\delta = \frac{dR}{L}$. This equation has exactly one negative root, with the other two either being positive real roots or being a conjugate pair of complex roots. The existence of two positive real roots requires that $\delta < \tilde{\delta}$, where $\tilde{\delta}$ is determined by the tangential equations

$$h(z) = h'(z) = 0 \tag{4.7}$$

for $z > 0$. Solving (4.7) we can obtain

$$108\,\delta^2 + \tilde{A}\delta - A^2(1-A)^2 = 0 \tag{4.8}$$

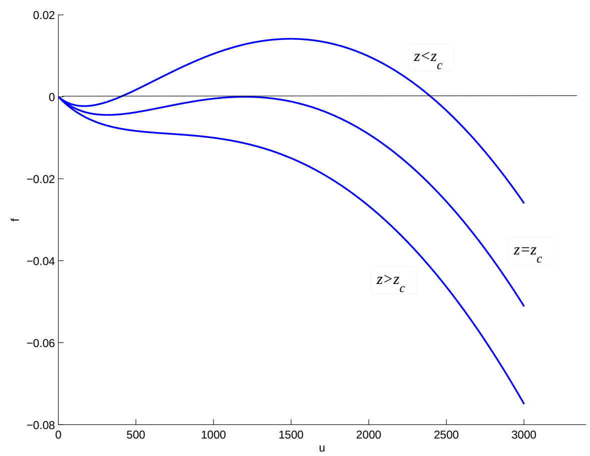

FIGURE 1. Curves of $f_0(u,z)$ as a function of u when $A = 0.2$, giving $z_c = 0.08$. Curves are shown for $z = 0.06$(upper), $z = 0.08$, and $z = 0.1$(lower).

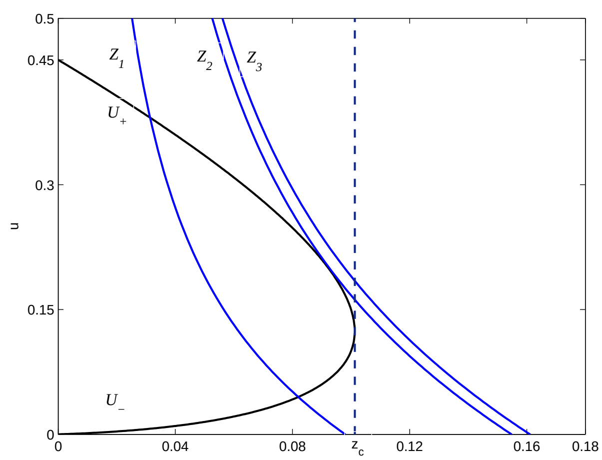

FIGURE 2. The nullclines for $f_0(u,z) = 0$ and $g_0(u,z) = 0$ on $z-u$ plane. Here $d = 1$, $A = 0.1$, $L = 1$. Thus $\tilde{R} = 0.0634$ and $z_c \approx 0.101$. U_+ and U_- are the curve described by (4.5); Z_1, Z_2, and Z_3 are the curves described by (4.3) for $R = 0.029 < \tilde{R}$, $R = \tilde{R}$ and $R = 0.07 > \tilde{R}$ respectively.

which has the positive root $\tilde{\delta}$ (also see [17] for details):

$$\tilde{\delta} = \frac{1}{216}\left(\sqrt{\tilde{A}^2 + 432A^2(1-A)^2} + \tilde{A}\right).$$

Thus, when $R < \tilde{R}$ (equivalently $\delta < \tilde{\delta}$), (4.6) has two positive roots, or in other words, the two curves described by (4.3) and (4.4) have two intersections for $z > 0$ on the z–u plane (see Figure 2). On the other hand, the equation $g_0(u, z) = 0$ can also be solved for $z > 0$ in terms of u, giving

$$z(u) = \frac{1}{4}\left(\sqrt{(A+2u)^2 + \frac{8dR}{L}} - (A+2u)\right). \tag{4.9}$$

Obviously, (4.9) describes the same curve as (4.3) does in the z–u plane.

Returning to the variable n by the relation $z = n + \frac{R}{2L}x^2$, we see that in the n–u plane the relation of intersections of $g(x, u, n) = 0$ and $f(x, u, n) = 0$ is qualitatively retained (in the sense of x dependent translations) as in n–z plane for that of $g_0(u, z) = 0$ and $f_0(u, z) = 0$. In particular, the above analysis shows that for every $u \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R})$, $g(x, u(x), n) = 0$ has a unique solution for n, denoting it by $n_u(x)$:

$$n_u(x) = \frac{1}{4}\left(\sqrt{(A+2u)^2 + \frac{8dR}{L}} - (A+2u)\right) - \frac{R}{2L}x^2, \tag{4.10}$$

Note that $g(x, u, n)$ is decreasing in n and $g(x, u, n_u) = 0$, by Theorem 3.1, for given $u \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R})$, there is a unique solution $n_u^*(x)$ to the problem

$$\begin{cases} dn_{xx} + g(x, u, n) = 0, & x \in \Omega, \\ \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{4.11}$$

satisfying $\inf_{\bar{\Omega}} n_u(x) \le n_u^*(x) \le \sup_{\bar{\Omega}} n_u(x)$ on $\bar{\Omega}$. From (4.10), we know that n_u is decreasing in u and $n_u(x) \to -\frac{R}{2L}x^2$ as $\inf_{\bar{\Omega}} u \to +\infty$. Hence, there exists a $u_c \in C^2(\bar{\Omega})$, such that for any $u > u_c$ on $\bar{\Omega}$, we have

$$n_u(x) < \hat{n}_c(x) \triangleq z_c - \frac{R}{2L}x^2.$$

This implies that $n_u^*(x) < \hat{n}_c(x)$ on $\bar{\Omega}$ when $\inf_{\bar{\Omega}} u$ is sufficiently large.

From the previous discussion, we know that as long as $R < \tilde{R}$, the two curves given by (4.4) and (4.10) always have two intersections in the n–u plane for every $x \in \bar{\Omega}$. By Theorem 3.2 and (4.10), we know $n_u^*(x) \to n_u(x) \to 0$ uniformly as $R \to 0$, which shows that when $R > 0$ is sufficiently small, $n_u^*(x)$ can intersects with $u_-(n + \frac{R}{2L}x^2)$ in the n-u plane at some $n > -\frac{R}{2L}x^2$ for all $x \in \bar{\Omega}$. This further implies that there is an $R_c \in (0, \tilde{R})$ such that when $R < R_c$, there exists a $u_0 > u_c$ satisfying not only $n_{u_0}^*(x) < \hat{n}_c(x)$ on $\bar{\Omega}$ but also $u_-(n_{u_0}^*(x) + \frac{R}{2L}x^2) > u_0$ on $\bar{\Omega}$ (see the definition of $u_-(z)$ in (4.5)).

Now, suppose $R < R_c$ and a u_0 is chosen such that $n_{u_0}^*(x) < \hat{n}_c(x)$ and $u_-(n_{u_0}^*(x) + \frac{R}{2L}x^2) > u_0$ on $\bar{\Omega}$. Let

$$\begin{aligned} \underline{n} &= -\frac{R}{2L}x^2, \ \ \bar{n} = n_{u_0}^*(x), \\ \underline{u} &= \sup_{\bar{\Omega}} u_-(n_{u_0}^*(x) + \frac{R}{2L}x^2), \\ \bar{u} &= \frac{1-A}{2} + \varepsilon. \end{aligned} \tag{4.12}$$

where ε is any positive number and hence $\underline{u}$, $\bar{u}$ are constants for fixed d, R, A, L. It is easy to check that $\underline{n} \le \bar{n}$ and $\underline{u} \le \bar{n}$. In fact, by the definition of $u_\pm(z)$ in (4.5), we have

$$\sup_{\bar\Omega} u_+(n^*_{u_0}(x) + \frac{R}{2L}x^2) \le \sup_{z\ge 0} u_+(z) = u_+(0) = \frac{1-A}{2}, \tag{4.13}$$

$$\inf_{\bar\Omega} u_+(n^*_{u_0}(x) + \frac{R}{2L}x^2) > \inf_{z\ge 0} u_+(z) = u_+(z_c) = \frac{1-A^2}{8}, \tag{4.14}$$

$$\sup_{\bar\Omega} u_-(n^*_{u_0}(x) + \frac{R}{2L}x^2) < \sup_{z\ge 0} u_-(z) = u_-(z_c) = \frac{1-A^2}{8}. \tag{4.15}$$

Thus,

$$\underline{u} < \inf_{\bar\Omega} u_+(n^*_{u_0}(x) + \frac{R}{2L}x^2) < \sup_{\bar\Omega} u_+(n^*_{u_0}(x) + \frac{R}{2L}x^2) < \bar u. \tag{4.16}$$

By the monotonicity of $g(x,u,n)$ and $f(x,u,n)$ and with some straightforward verifications, we can see that $\underline{n}$, $\bar n$ $\underline{u}$ and $\bar u$ satisfy

$$\begin{cases} -\bar u_{xx} - f(x,\bar u,\underline{n}) \ge 0 \ge -\underline{u}_{xx} - f(x,\underline{u},\bar n), \\ -d\bar n_{xx} - g(x,\underline{u},\bar n) \ge 0 \ge -d\underline{n}_{xx} - g(x,\bar u,\underline{n}), \\ \dfrac{\partial \bar u}{\partial \nu}|_{\partial\Omega} \ge 0, \dfrac{\partial \bar n}{\partial \nu}|_{\partial\Omega} \ge 0, \\ \dfrac{\partial \underline{u}}{\partial \nu}|_{\partial\Omega} \le 0, \dfrac{\partial \underline{n}}{\partial \nu}|_{\partial\Omega} \le 0. \end{cases} \tag{4.17}$$

We only show the verifications of two of the inequalities in (4.17), as the rest are similar. By the first inequality in (4.16) and $n^*_{u_0}(x) < \hat n_c(x)$, we have

$$\begin{aligned} -\underline{u}_{xx} - f(x,\underline{u},\bar n) &= -f(x, \sup_{\bar\Omega} u_-(n^*_{u_0}(x) + \frac{R}{2L}x^2), n^*_{u_0}(x)) \\ &< -f(x, u_-(n^*_{u_0}(x) + \frac{R}{2L}x^2), n^*_{u_0}(x)) \\ &= 0. \end{aligned} \tag{4.18}$$

We have seen that $\underline{u} > u_0$; hence, by the monotone property of $g(x,u,n)$ with respect to u, we have

$$\begin{aligned} -d\Delta\bar n - g(x,\underline{u},\bar n) &= -d\Delta n^*_{u_0}(x) - g(x, \sup_{\bar\Omega} u_-(n^*_{u_0}(x) + \frac{R}{2L}x^2), n^*_{u_0}(x)) \\ &> -d\Delta n^*_{u_0}(x) - g(x, u_0, n^*_{u_0}(x)) \\ &= 0. \end{aligned} \tag{4.19}$$

In other words, $(\underline{u},\underline{n})$ is a lower solution of (4.1) and $(\bar u,\bar n)$ is an upper solution of (4.1). Now, by [21], System (4.1) has at lease one non-constant solution $(u^*(x), n^*(x))$ satisfying

$$\underline{u} \le u^*(x) \le \bar u,$$
$$\underline{n} \le n^*(x) \le \bar n.$$

The proof is completed. □

From the proof of Theorem 4.1, we have the following non-existence result. In this case, the *global* stability of the unique steady state E_0 is obtained as a consequence.

Theorem 4.2. *For given parameters A and L, let*

$$d_+ = \frac{L^2}{4}\left((1-A)^2 + 2 + (1-A)\sqrt{(1-A)^2+4}\right), \tag{4.20}$$

$$R^c_\pm = \frac{1}{4L^3}\left(4d - (A^2-2A+3)L^2 \pm 2\sqrt{L^4 - 2(A^2-2A+3)dL^2 + 4d^2}\right). \tag{4.21}$$

If

(H1) $d > d_+$;
(H2) $R \in (R^c_-, R^c_+)$,

then there is no coexistence steady state to (1.5); consequently, the fertile-free steady state E_0 is the unique steady state of (1.5) and it is globally asymptotically stable.

Proof. From the definition of $n_u(x)$ in (4.10), we know $n_u(x)$ is decreasing with respect to u. So by inequality (4.13),

$$\inf_{\bar\Omega} n_u(x) \geq n_{\frac{1-A}{2}}(L) = \frac{1}{4}\sqrt{1+\frac{8dR}{L}} - \frac{1}{4} - \frac{RL}{2}. \tag{4.22}$$

The assumptions (H1) and (H2) imply that $n_{\frac{1-A}{2}}(L) > \sup_{\bar\Omega} n_c = \frac{(1-A)^2}{8}$. In fact,

$$n_{\frac{1-A}{2}}(L) > \sup_{\bar\Omega} n_c \tag{4.23}$$

$$\Leftrightarrow 2\sqrt{1+\frac{8dR}{L}} > (1-A)^2 + 2 + 4RL \tag{4.24}$$

$$\Leftrightarrow 16L^2R^2 + \frac{8}{L}\left(2L^2 + (1-A)^2L^2 - 4d\right)R + (1-A)^4 + 4(1-A)^2 < 0. \tag{4.25}$$

We regard the left hand side of (4.25) as a quadratic function of R. Then assumption (H1) guarantees that the discriminant of this quadratic function is positive, implying that the quadratic has two real roots $R^c_- < R^c_+$; and assumption (H2) implies the inequality (4.25) hold.

From Theorem 3.1, if $(\hat u(x), \hat n(x))$ is a solution of (4.1) with $\hat u(x) \not\equiv 0$, it must satisfy $\hat n(x) = n^*_{\hat u}(x)$, and hence $\inf_{\bar\Omega} n_{\hat u}(x) \leq \hat n(x) \leq \sup_{\bar\Omega} n_{\hat u}(x)$ on $\bar\Omega$. When (H1) and (H2) hold, the vertical line $n_{\frac{1-A}{2}}(L)$ in n-u plane does not intersect with the curve $u_\pm(n + \frac{R}{2L}x^2)$ for any $x \in \bar\Omega$. So from (4.22), $\inf_{\bar\Omega} n_{\hat u}(x)$ does not intersect with the curve $u_\pm(\hat n + \frac{R}{2L}x^2)$ for any $x \in \bar\Omega$. This implies that $f(x, \hat u(x), \hat n(x)) < 0$ on $\bar\Omega$, a contradiction to

$$\int_\Omega f(x, \hat u(x), \hat n(x))dx = -\int_\Omega \hat u_{xx} dx = 0.$$

Therefore, there can not be a coexistence steady state solution to (4.1), and consequently, E_0 is the unique steady state for (1.5) under assumptions (H1) and (H2).

Next we show that when (H1) and (H2) hold, E_0 is indeed globally asymptotically stable. Noting that system (1.5) is a two dimensional competitive system, it can be viewed as a monotone dynamical system with respect to the partial ordering $\leq_K$ induced by the second quadrant cone:

$$K = \{(u, n) \in X : u \leq 0,\ n \geq 0\}$$

that is,

$$(u_1, n_1) \leq_K (u_2, n_2) \Leftrightarrow u_1 \geq u_2 \text{ and } n_1 \leq n_2.$$

Alternatively, by the change of the variables $v = -u$, system (1.5) is transformed into

$$\begin{cases} v_t = v_{xx} - v\left(\dfrac{v}{n + \frac{R}{2L}x^2 - v} + A + 2(n + \frac{R}{2L}x^2 - v)\right), & x \in \Omega, \\ n_t = dn_{xx} - A(n + \frac{R}{2L}x^2) - 2(n + \frac{R}{2L}x^2)(n + \frac{R}{2L}x^2 - v) + \frac{dR}{L}, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \ \dfrac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = 0, & t > 0, \\ v(x,0) = v_0(x) \le 0, \ n(x,0) = n_0(x) \ge -\frac{R}{2L}x^2, & x \in \Omega, \end{cases} \tag{4.26}$$

which is a cooperative system. The invariant set X for (1.5) is obviously transformed to the invariant set X' for (4.26) where

$$X' = \left\{(v,n) \in Y | v(x) \le 0, n(x) \ge -\frac{R}{2L}x^2, x \in \bar{\Omega}\right\}.$$

By Theorem 2.1, we conclude that system (4.26) generates a monotone semiflow Ψ_t on X' with respect to the natural ordering (that is, the ordering induced by the first quadrant).

We now show that Ψ_t is actually strongly monotone on X'. To this end, we let ϕ, $\psi \in X'$ with $\phi < \psi$. Let $(v(t,x,\phi), n(t,x,\phi)) = (\Psi_t\phi)(x)$ and $(v(t,x,\psi), n(t,x,\psi)) = (\Psi_t\psi)(x)$ be the respective solutions of (4.26) corresponding to these two initial functions. Denote by $F(x,v,n)$ and $G(x,v,n)$ the two nonlinear functions on the right hand side of (4.26). Note that the Jacobian matrix

$$J \triangleq \begin{pmatrix} F_v(x,v,n) & F_n(x,v,n) \\ G_v(x,v,n) & G_n(x,v,n) \end{pmatrix}$$

becomes reducible when $v = 0$ and $x = 0$, or when $n = \frac{-R}{2L}x^2$. However, it is easy to check that the matrix

$$h(t,x) = \begin{pmatrix} h_{11}(t,x) & h_{12}(t,x) \\ h_{21}(t,x) & h_{22}(t,x) \end{pmatrix}$$

is cooperative and irreducible for all $t > 0$ and $x \in \Omega$, where

$$h_{11}(x,t) = \int_0^1 \frac{\partial F}{\partial v}(x, sv(x,t,\phi) + (1-s)v(x,t,\psi), sn(x,t,\phi) + (1-s)n(x,t,\psi))ds,$$

$$h_{12}(x,t) = \int_0^1 \frac{\partial F}{\partial n}(x, sv(x,t,\phi) + (1-s)v(x,t,\psi), sn(x,t,\phi) + (1-s)n(x,t,\psi))ds,$$

$$h_{21}(x,t) = \int_0^1 \frac{\partial G}{\partial v}(x, sv(x,t,\phi) + (1-s)v(x,t,\psi), sn(x,t,\phi) + (1-s)n(x,t,\psi))ds,$$

$$h_{22}(x,t) = \int_0^1 \frac{\partial G}{\partial n}(x, sv(x,t,\phi) + (1-s)v(x,t,\psi), sn(x,t,\phi) + (1-s)n(x,t,\psi))ds.$$

Therefore, by a similar argument to that in the proof of [26, Theorem 7.4.1], we conclude that

$$v(t,x,\phi) < v(t,x,\psi), \ \text{ and } \ n(t,x,\phi) < n(t,x,\psi) \ \text{ for all } x \in \Omega \ \text{ and } \ t > 0.$$

Thus, Ψ_t is strongly monotone on X' and hence, strongly order preserving. Translating this conclusion in terms of original system (1.5), we know that the solution semiflow Φ_t of (1.5) is strongly order preserving with respect to the ordering $\le_K$. Combining this with the [26, Theorem 2.3.1], we conclude that the unique steady state E_0 is globally asymptotically stable, completing the proof. ∎

5. **Conclusion and discussion.** In this paper, we have considered a reaction diffusion system to model the the Sterile Insect Releasing Method (SIRM). The model is an alternation of the models considered in [17] and [14], in that the spatial domain is bounded and the releasing is only on the boundary. We have proved the well-posedness of this alternated model, and obtained conditions under which the fertile-free steady state exists and is globally asymptotically stable, accounting for success of the SIRM. We have also obtained conditions under which coexistence steady state exists, corresponding to failure of the SIRM. These results show that, the SIRM with releasing only on the boundary can also successfully eradicate the fertile insects provided that the release strength is sufficient large.

We point out that (H1) and (H2) are just sufficient conditions to exclude existence of coexistence steady state. Extensive numerical simulations show there is a larger range for the parameters for which (H1) and (H2) do not hold, but the fertile-free steady state is also globally asymptotically stable (hence there is no coexistence steady state). Theoretically seeking conditions weaker than (H1)-(H2) is desirable and constitutes a good yet challenging mathematical problem.

The aforementioned conditions are related to the calculated parameters d_+ and R_-^c. Exploring the dependence of these two calculated parameters on the model parameters can reveal some biological implications. For example, straightforward calculation show that $\frac{\partial R_-^c}{\partial L} > 0$. This together with (H2) implies that the larger the domain is, the more it costs (lager R is needed) to eradicate the fertile insect.

Acknowledgments. The authors are grateful to the referee and the handling editor for their valuable comments which have led to an improvement in the presentation of the paper. This work was completed when the first author was visiting the University of Western Ontario and he would like to thank the staff in the Department of Applied Mathematics for their help and thank the university for its excellent facilities and support during his stay.

REFERENCES

[1] H. J. Barclay, *The sterile release method for population control with interspecific competition*, Res. Pop. Ecol., **23** (1981), 145–155.

[2] H. J. Barclay, *Models for pest control: Complementary effects of periodic release of sterile pests and parasitoids*, Theor. Pop. Biol., **32** (1987), 76–89.

[3] H. J. Barclay, *Demographic consequences of monogamy and polygamy for a sterile release program*, Protection Ecol., **6** (1984), 209–217.

[4] H. J. Barclay and M. Mackauer, *The sterile insect release method for pest control: A density dependent model*, Envir. Entom., **9** (1980), 810–817.

[5] H. J. Barclay and M. Mackauer, *Effects of sterile insect releases on a population under predation or parasitism*, Res. Pop. Ecol., **22** (1980), 136–146.

[6] A. H. Baumhover, A. J. Graham, B. A. Bitter, D. E. Hopkins, W. D. New, F. H. Dudley and R. C. Bushland, *Screwworm control though release of sterilized flies*, J. Econ. Entomol., **48** (1955), 462–466.

[7] A. A. Berryman, *Mathematical description of the sterile male principle*, Can. Entomol., **99** (1967), 858–865.

[8] K. Dietz, *The effect of immigration on genetic control*, Theor. Popul. Biol., **9** (1976), 58–67.

[9] Y. H. Du, "Order Structure and Topological Methods in Nonlinear PDEs. Vol. 1. Maximum Principle and Applications," World Scientific, Singapore, 2005.

[10] H. M. Graham, *Sterile pink bollworm: Field release for population suppression*, J. Econ. Entomol., **71** (1978), 233–235.

[11] G. W. Harrison, H. J. Barclay and P. van den Driessche, *Analysis of a sterile release model with predation*, J. Math. Biol., **16** (1982/83), 33–48.

[12] Y. Ito, *A model of sterile insect release for eradication of the melon fly*, Dacus cucurbitae COQUILLETT. Appl. Ent. Zool., **12** (1977), 303–310.

[13] O. Iwahashi, *Eradication of the melon fly, Dacus eucurbitae, from Kume Is. Okinawa. with the sterile insect release method*, Res. Popul. Ecol., **19** (1977), 87–98.

[14] W. Jiang, X. Li and X. Zou, *On a reaction-diffusion model for sterile insect release method in a bounded domain*, submitted.

[15] E. F. Knipling, *Possibilities of insect control or eradication through the use of sexually sterile males*, J. Econ. Entomol., **48** (1955), 459–462.

[16] E. S. Krafsur, H. Townson, G. Davidson and C. F. Curtis, *Srewworm eradication is what it seems*, Nature, **323** (1986), 495–496.

[17] M. A. Lewis and P. van Den Driessche, *Waves of extinction from sterile insect release*, Math. Biosci., **5** (1992), 221–247.

[18] A. W. Lindquist, *The use of gamma radiation for control or eradication of the Screw-worm*, J. Econ. Entomol., **48** (1955), 467–469.

[19] V. S. Manoranjan and P. van den Driessche, *On a diffusion model for sterile insect release*, Math. Biosci., **79** (1986), 199–208.

[20] D. R. Miller and D. E. Weidhaas, *Equilibrium populations during a sterile release program*, Environ. Entomol., **3** (1974), 211–216.

[21] C. V. Pao, "Nonlinear Parabolic and Elliptic Equations," Plenum Press, New York, 1992.

[22] R. E. Plant and R. T. Cunningham, *Analysis of the dispersal of sterile Mediterranean fruit flies (Diptera: Tephritidae) released from a point source*, Environ. Entomol., **20** (1991), 1493–1530.

[23] R. E. Plant and M. Mangel, *Modeling and simulation in agricultural pest management*, SIAM Rev., **29** (1987), 235–261.

[24] T. Prout, *The joint effects of the release of sterile males and immigration of fertilized females on a density regulated population*, Theor. Popul. Biol., **13** (1978), 40–71.

[25] M. D. Proverbs, J. R. Newton and D. M. Logan, *Codling moth control by the sterility method in twenty-one British Columbia orchards*, J. Econ. Entomol., **70** (1977), 667–671.

[26] H. L. Smith, "Monotone Dynamical System: An introduction to the Theory of Competitive and Cooperative Systems," Mathematical Surveys and Monographs, **41**, AMS, Providence, RI, 1995.

Received February 2012; 1st revision April 2012; 2nd revision April 2012.

E-mail address: lixinbox1984@gmail.com
E-mail address: xzou@uwo.ca