DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **24**, Number **10**, October **2019**

doi:10.3934/dcdsb.2019076

pp. **5633–5671**

SPATIAL DYNAMICS OF A LOTKA-VOLTERRA MODEL WITH A SHIFTING HABITAT

Yueding Yuan

School of Mathematics and Statistics, Central South University
Changsha 410083, Hunan, China
and
School of Mathematics and Computer Sciences, Yichun University
Yichun 336000, Jiangxi, China

Yang Wang

Department of Applied Mathematics, Western University
London, Ontario, N6A 5B7, Canada

Xingfu Zou*

School of Mathematics and Statistics, Central South University
Changsha 410083, Hunan, China
and
Department of Applied Mathematics, Western University
London, Ontario, N6A 5B7, Canada

(Communicated by Yuan Lou)

Abstract. In this paper, we study a Lotka-Volterra competition-diffusion model that describes the growth, spread and competition of two species in a shifting habitat. Our results show that (I) if the competition between the two species are either mutually strong or mutually weak against each other, the spatial dynamics mainly depend on environment worsening speed c and the spreading speed of each species in the absence of the other in the best possible environment; (II) if one species is a strong competitor and the other is a weak competitor, then the interplay of the species' competing strengths and the spreading speeds also has an effect on the spatial dynamics. Particularly, we find that a strong but slower competitor can co-persist with a weak but faster competitor, provided that the environment worsening speed is not too fast.

1. **Introduction.**

It is well known that spatial heterogeneity and diffusion play an important role when considering the interaction of biological species that can diffuse in the real world (see, e.g., [5, 6, 7, 16, 26, 34, 35]). Such a role can be well demonstrated even by diffusive Lotka-Volterra system for two biological species

2010 *Mathematics Subject Classification.* Primary: 92D40, 92D25; Secondary: 35K57, 93C10.

Key words and phrases. Climate change, competition, reaction-diffusion, Lotka-Volterra model, coexistence, shifting habitat, spreading speed.

Research was partially supported by National Natural Science Foundation of China (No. 11561068) and China Postdoctoral Science Foundation (2016M592442) and NSERC of Canada (No. RGPIN-2016-04665).

* Corresponding author.

or two strains of the same species. For example, when studying the evolution of dispersals, Hastings [15] and Dockery et al [9] considered the following R-D system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x) - u_1 - u_2], \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x) - u_1 - u_2], \end{cases} \quad t > 0, \ \ x \in \Omega \subset \mathbb{R}^m, \tag{1}$$

where $u_1(t, x)$ and $u_2(t, x)$ represent the populations of two competing species with respective diffusion rates d_1 and d_2. Here, because the goal is to see whether slower or faster diffusion will have a selection advantage, the authors assumed that all biological characteristics of the two species are the same except for the diffusion rates ($d_1 \neq d_2$). A scenario for such a case is that one species (or strain) is mutated from the other with different diffusion rate. Hence, in (1), the two species share the same competition strength for each against the other (normalized to 1) and the same growth rate $r(x)$ which reflects the intrinsic production and the habitat environment (e.g., richness of resources and quality of the living habitat etc.)

It turns out that if Ω is a bounded set and (1) is associated with the zero flux boundary condition, then the species with slower diffusion rate will win the competition, e.g., if $d_1 < d_2$, then all positive solutions of (1) converge to $(u_1^*(x), 0)$ where $u_1^*(x)$ is the unique positive solution of the boundary value problem

$$\begin{cases} d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x) - u_1], \ \ x \in \Omega, \\ \dfrac{\partial u_1}{\partial n} = 0, \quad x \in \partial\Omega \end{cases} \quad t > 0, \ \ x \in \Omega \subset \mathbb{R}^m. \tag{2}$$

When allowing different competition strengths between the two species, the Lotka-Volterra system (1) is modified to

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x) - u_1 - a_1 u_2], \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x) - a_2 u_1 - u_2], \end{cases} \quad t > 0, \ \ x \in \Omega \subset \mathbb{R}, \tag{3}$$

where the constant $a_1 > 0$ ($a_2 > 0$) represents the competition strength of species 2 (species 1) against species 1 (species 2). Now for such a slightly more general model (than (1)), each species faces choices in dispersal rate and local competition strength, and the interplay of the two sets of parameters $\{d_1, d_2\}$ and $\{a_1, a_2\}$ has revealed some very interesting and surprising results on the asymptotic behaviours of the solutions, including the possibility of a globally asymptotically stable positive (co-persistent) steady state. For details, see, e.g., [9, 15, 17, 18, 19, 23, 24, 25, 31, 32] and the references therein.

On the other hand, the climate change in recent years has been a major concern of the scientific community, including ecologists and applied mathematicians, see, e.g., [2, 3, 4, 14, 20, 27, 29, 30, 33, 37, 38, 39, 40, 41, 42, 45, 50]. A simple climate change pattern is the shifting of environment quality with a constant speed. This would translate to the shifting of the habitat quality which would be reflected by the shifting of the growth rate for a species. With such a consideration, Li et al [27] considered the following R-D equation for a single species living in the 1-dimensional whole space $\mathbb{R}$:

$$\frac{\partial u_1}{\partial t} = d \frac{\partial^2 u_1}{\partial x^2} + u_1[r(x - ct) - u_1], \quad t > 0, \ x \in \mathbb{R}. \tag{4}$$

Here, the way the growth rate function r depends on the time and location is through a moving pattern with a constance speed c, and $r(\cdot)$ is assumed to satisfy

(A) $r(x)$ is continuous, nondecreasing, bounded and piecewise continuously differentiable for all $x \in \mathbb{R}$ with $0 < r(\infty) < \infty$ and $-\infty < r(-\infty) < 0$.

The non-decreasing property of $r(x)$ assumes that the environment gets worse as time goes, and the negativity of $r(-\infty)$ accounts for a scenario that the environment is shifting to a very severe level. It is shown in [27] that if the environment's worsen speed $c > c^*(\infty) := 2\sqrt{dr(\infty)}$, then the species will go to extinction in the habitat; while when $c < c^*(\infty)$, then the species satisfying some condition on its initial distribution will persist and spread along the gradient of the shifting habitat at an asymptotic spreading speed being precisely c^*.

The feature of "shifting with constant speed" represented by the moving frame, allows one to explore the traveling wave solutions of the form $u(t,x) = U(x - ct) = U(\xi)$ governed by a second order non-autonomous ODE with the moving coordinate $\xi = x - ct$ as the independent variables. In this regards, for the case $c > c^*(\infty)$, by proving the existence of extinction wave solutions, Hu and Zou [21] confirmed that the extinction indeed occurs for (4) through the form of waves with the constant speed c. If the $-\infty < r(-\infty) < 0$ is replaced by $r(-\infty) > 0$ in **(A)**, the biological scenario is changed from "severely worsening environment" to "mildly worsening environment". The recent work [3] has investigated traveling waves for a general KPP type reaction diffusion equation with mildly worsening environment that includes (4) as a special case. The much earlier work [2] also explored the traveling waves of a general R-D equation that has (4) as special case but with the growth function $r(\xi)$ *having support only on an finite interval*. When considering a pathogen's population under the shifting host population, Fang *et al* [10] also obtained a scalar equation of the form (4) satisfying **(A)**. In addition to spreading speed of the pathogen in comparison the spread speed of host, they also discussed the existence of traveling wave front solutions, leading to a better understanding of spatial-temporal patterns under such a shifting environment.

When two competing species face a shifting environment represented by $r(x-ct)$ as in Li et al (4), the model system (3) is naturally revised to

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x-ct) - u_1 - a_1 u_2], \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x-ct) - a_2 u_1 - u_2], \end{cases} \quad t > 0, \quad x \in \Omega \subset \mathbb{R}. \tag{5}$$

Now there arises a question: when a species not only faces an environment worsening represented by the shifting pattern given by $r(x - ct)$ with r satisfying the assumption (A), but also encounters a competition from another species, how should the species choose its diffusion strategy? On the one hand, in light of the results in Li et al [27], faster diffusion (so that $c^*(\infty) = 2\sqrt{d\,r(\infty)} > c$) would help the species to escape the environment worsening and thereby, help the species to persist. On the other hand, in view of the results in Hastings [15] and Dockery et al [9], slower diffusion should be favoured. It seems that there should be some strategy for a species which can balance the pressure from environment shifting and competition so that the species can persist. It is also possible that if the two species diffuse *improperly*, they will both go to extinction, whereas if they diffuse *properly*, they can both persist in some manner, surviving the environmental shifting and the competition. Exploring conditions for each of the above possibilities, particularly conditions for co-persistence, constitutes the goal of of this paper. We will achieve this goal by studying the spatial dynamics of model (5), with particular interest in conditions for the two species to co-persist. Note that model (5) is heterogeneous in space and time. The heterogeneity described by $r(x-ct)$ makes the existing theory and results for the case of constant r (see, e.g., [8, 28] and the reference therein) not applicable to (5). In order to prove our main result, for the case of *weak competition* meaning that $0 < a_j < 1$ for $j = 1, 2$, we will use the fluctuation method, developed in Li et al. [27], and for the case of *strong competition* meaning that $a_j \geq 1$ for $j = 1, 2$, we will develop an alternative approach which enables us to obtain the spatial dynamics of model (5) in such a case.

Under assumption (A) and in the case of either weak competition or strong competition, the limit system of (5) has a positive co-persistence (positive) constant steady state (u_1^+, u_2^+) where

$$u_i^+ = \frac{(1-a_i)r(\infty)}{1-a_1a_2}, \quad i = 1, 2.$$

We show that the spatial dynamics mainly depend on c, $c_i^*(\infty) = 2\sqrt{d_i r(\infty)}$ and $\hat{c}_i^*(\infty) = \sqrt{1-a_i}c_i^*(\infty)$, $i = 1, 2$, and $\hat{c}^*(\infty) = \min\{\hat{c}_1^*(\infty),\ \hat{c}_2^*(\infty)\}$. Without loss of generality, we always assume that $d_1 < d_2$, hence $c_1^*(\infty) < c_2^*(\infty)$. We will show that (i) if $c > c_2^*(\infty)$, then two competing species will both go extinct in the habitat; (ii) if $c_1^*(\infty) < c < c_2^*(\infty)$, then species 1 will become extinct in the habitat and species 2 will persist and spread; (iii) if $0 < c < c_1^*(\infty)$ and $a_j \geq 1$ for $j = 1, 2$, then species 1 will become extinct in the habitat and species 2 will persist and spread; (iv) if $0 < c < \hat{c}^*(\infty)$ and $0 < a_j < 1$ for $j = 1, 2$, then two competing species will coexist.

The rest of this paper is organized as follows. In Section 2, we present the main mathematical results regarding the spatial dynamics of (5). In Section 3 we give some numerical simulation results that help illustrate the results from Section 2 and motivate some conjectures. Section 4 contains some discussion of the results and comparison with some recent work. To make the reading smoother, we leave the proof of Lemma 2.5 to the Appendix.

2. **Mathematical results.**

In this section, we present mathematical results on the spatial dynamics of model (5) for the casen $\Omega = \mathbb{R}$. We first introduce some notations. Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all reals and nonnegative reals, respectively. For any $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}_+^2$, we write $u \leq v$ if $u_1 \leq v_1$ and $u_2 \leq v_2$. For any constant k, we denote by $\vec{k}$ the vector (k, k). Define $F_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}$, $i = 1, 2$, by

$$F_1(t, x, u) = u_1[r(x-ct) - u_1 - a_1u_2]$$

and

$$F_2(t, x, u) = u_2[r(x-ct) - a_2u_1 - u_2]$$

for any $(t, x, u) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2$. Then we may rewrite (5) as the following more convenient form with given non-negative initial function:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = d_1\dfrac{\partial^2 u_1(t,x)}{\partial x^2} + F_1(t, x, u(t,x)),\ t > 0,\ x \in \mathbb{R}, \\ \dfrac{\partial u_2(t,x)}{\partial t} = d_2\dfrac{\partial^2 u_2(t,x)}{\partial x^2} + F_2(t, x, u(t,x)),\ t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \phi(x) \geq 0,\ \ x \in \mathbb{R}. \end{cases} \tag{6}$$

Throughout the paper, we assume that the function r satisfies the assumption (A) and $d_1 < d_2$, implying $c_1^*(\infty) < c_2^*(\infty)$.

For any $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, $u = (u_1, u_2)$, $v = (v_1, v_2)$ with u_i, $v_i \in [0, r(\infty)]$ for $i = 1, 2$, one can easily verify the following Lipschitz condition:

$$|F_1(t,x,u_1,u_2) - F_1(t,x,v_1,v_2)| \le 4r(\infty)|u_1 - v_1| + r(\infty)|u_2 - v_2|$$

and

$$|F_2(t,x,u_1,u_2) - F_2(t,x,v_1,v_2)| \le r(\infty)|u_1 - v_1| + 4r(\infty)|u_2 - v_2|.$$

Clearly, if $\vec{0} \le \phi(x) \le \vec{r}(\infty)$, then $\widehat{u} \equiv \vec{0}$ and $\widetilde{u} \equiv \vec{r}(\infty)$ are coupled upper and lower solutions of (6) (see [36] for definition), where $\vec{0} = (0,0)$, $\vec{r}(\infty) = (r(\infty), r(\infty))$. The theory on the existence and uniqueness of solutions for reaction-diffusion systems has been well established (e.g.,Theorem 2.1 in [36]), by which, it is known that the initial value problem (6) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$ has a unique classical solution $u(t,x)$ with $\vec{0} \le u(t,x) \le \vec{r}(\infty)$.

For convenience of notations, we follow [27] to introduce some related functions. For $r(x) > 0$, define

$$c_i^*(x) = 2\sqrt{d_i r(x)}, \quad i = 1, 2.$$

It is easily seen that

$$c_i^*(x) = \inf_{\mu > 0} h_i(x;\mu),$$

where

$$h_i(x;\mu) = \frac{d_i \mu^2 + r(x)}{\mu}, \quad i = 1, 2.$$

The infimums occur at $\mu_i^*(x) = \sqrt{r(x)/d_i}$, $i = 1, 2$. By [27], $c_i^*(\infty)$ is nothing but the asymptotic spread speed for species i in the absence of species j $(j \ne i)$. Next, we focus on three case: $c > c_2^*(\infty)$, $c_1^*(\infty) < c < c_2^*(\infty)$ and $0 < c < c_1^*(\infty)$.

The following result shows that if the two species initially live only on a bounded domain and their respective spreading speed is less than the habitat's worsening speed c, then both species will go to extinction, regardless of the competition strength.

Theorem 2.1. *Assume that (A) holds and $c > c_2^*(\infty)$. Let $u(t,x,\phi)$ be the solution of (6) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$. If $\phi(x) \equiv \vec{0}$ for all sufficiently large x, then for every $\varepsilon > 0$ there exits $T > 0$ such that $u(t,x,\phi) \le \vec{\varepsilon}$ for all $(t,x) \in [T,\infty) \times \mathbb{R}$, where $\vec{\varepsilon} = (\varepsilon, \varepsilon)$.*

Proof. Let $\overline{u}(t,x) \equiv (\overline{u}_1(t,x), \overline{u}_2(t,x))$ be the solution of the following decoupled system

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = d_1 \dfrac{\partial^2 u_1(t,x)}{\partial x^2} + u_1(t,x)[r(x-ct) - u_1(t,x)], \ t > 0, \ x \in \mathbb{R}, \\ \dfrac{\partial u_2(t,x)}{\partial t} = d_2 \dfrac{\partial^2 u_2(t,x)}{\partial x^2} + u_2(t,x)[r(x-ct) - u_2(t,x)], \ t > 0, \ x \in \mathbb{R}, \\ u(0,x) = \phi(x), \ x \in \mathbb{R}. \end{cases} \tag{7}$$

Since $F_i(t,x,u) \le u_i(r(x-ct) - u_i)$, $i = 1, 2$, we have $u(t,x,\phi) \le \overline{u}(t,x)$ for all $t > 0$ and $x \in \mathbb{R}$. Thus, the conclusion of the theorem follows from Theorem 2.1 in [27]. □

The next result indicates that if one of the species spreads with a speed faster than the habitat's worsening speed, then that species is able to persist in the spreading sense, as long as that species' initial presence is significant in a very mild sense as stated in the following theorem.

Theorem 2.2. *Assume that (A) holds and $c_1^*(\infty) < c < c_2^*(\infty)$. Let $u(t,x,\phi)$ be the solution of (6) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$. Then the following statements hold.*

(i) *If $\phi_1(x) \equiv 0$ for all sufficiently large x, then for every $\varepsilon > 0$ there exits $T > 0$ such that $u_1(t,x,\phi) \le \varepsilon$ for all $(t,x) \in [T,\infty) \times \mathbb{R}$;*

(ii) *For any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x\le t(c-\varepsilon)} u_2(t,x,\phi)\right] = 0;$$

(iii) *If $\phi_2(x) \equiv 0$ for all sufficiently large x, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x\ge t(c_2^*(\infty)+\varepsilon)} u_2(t,x,\phi)\right] = 0;$$

(iv) *If $\phi_1(x) \equiv 0$ for all sufficiently large x and $\phi_2(x) > 0$ on a closed interval, then for any ε with $0 < \varepsilon < (c_2^*(\infty) - c)/2$,*

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c_2^*(\infty)-\varepsilon)} |r(\infty) - u_2(t,x,\phi)|\right] = 0.$$

Proof. Part (i)-(iii) follow from the same comparison argument as in the proof of Theorem 2.1, together with the corresponding results in [27]. To prove (iv), we note that by (ii), for any $\delta \in (0, r(\infty)/2)$, there exists $T > 0$ such that $u_1(t,x,\phi) \le \delta$, and hence, $u_2(r(x-ct) - u_2 - a_2\epsilon) \le F_2(t,x,u) \le u_2(r(x-ct) - u_2)$, for $t \ge T$. Then, a comparison argument leads to $\underline{u}_2(t,x) \le u_2(t,x,\phi) \le \overline{u}_2(t,x)$ for all $(t,x) \in [T,\infty) \times \mathbb{R}$, where $\underline{u}_2(t,x)$ and $\overline{u}_2(t,x)$ are the solutions of

$$\begin{cases} \dfrac{\partial u_2(t,x)}{\partial t} = d_2\dfrac{\partial^2 u_2(t,x)}{\partial x^2} + u_2(t,x)(r_\delta(x-ct) - u_2(t,x)),\ t > T,\ x \in \mathbb{R}, \\ u_2(T,x) = u_2(T,x,\phi),\ x \in \mathbb{R} \end{cases} \tag{8}$$

and

$$\begin{cases} \dfrac{\partial u_2(t,x)}{\partial t} = d_2\dfrac{\partial^2 u_2(t,x)}{\partial x^2} + u_2(t,x)(r(x-ct) - u_2(t,x)),\ t > T,\ x \in \mathbb{R}, \\ u_2(T,x) = u_2(T,x,\phi),\ x \in \mathbb{R}, \end{cases} \tag{9}$$

respectively, where $r_\delta(x-ct) = r(x-ct) - a_2\delta$. If $\phi_2(x) > 0$ on a closed interval, then $u_2(T,x,\phi) > 0$ on this closed interval, and thus, part (iv) follows at once from (8), (9), the arbitrariness of δ and Theorem 2.2 of [27]. □

If $c < c_1(\infty)$, then obviously (ii) and (iii) also holds for species 1, as stated in the following theorem.

Theorem 2.3. *Assume that (A) holds and $0 < c < c_1^*(\infty)$. Let $u(t,x,\phi)$ be the solution of (6) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$. Then the following two statements are valid.*

(i) *For any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x\le t(c-\varepsilon)} u_i(t,x,\phi)\right] = 0, \quad i = 1,2;$$

(ii) *If $\phi(x) \equiv \vec{0}$ for all sufficiently large x, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x\geq t(c_i^*(\infty)+\varepsilon)} u_i(t,x,\phi)\right] = 0, \quad i = 1,2.$$

Theorem 2.3 shows that if the spread speeds for the two species are *both larger* than the habitat's worsening speed (i.e., $c < c_1^*(\infty)$ due to the assumption $d_1 < d_2$), then an observer moving toward the right direction with a speed less than the habitat's worsening speed c, or moving with a speed larger than c but with the two species initially living only in a bounded domain, will not be able see individuals of the two species as $t \to \infty$. In such a case, it is very natural and interesting to ask if the conclusion (iv) in Theorem 2.2 remains true for *species 1 as well* when $c < c_1^*(\infty)$, meaning that species 1 can also persist in the moving mode as stated in Theorem 2.2-(iv) for species 2. This problem turns out to be very challenging due to the presence of competition between the two species. In the sequel, we study two cases: the case of week competition and the case involving strong competition. In the case of *week competition*, we shall show that under a stronger condition ($\hat{c}^*(\infty) > c$), the answer to the question is *affirmative*, with the persistence levels for each species modified to reflect the effect of competition (see Theorem 2.7). But, in the case involving *strong competition*, the answer to the question is *negative*, with species 1 becoming extinct in the habitat and species 2 persisting and spreading (see Theorem 2.11). To proceed toward the goal, the rest of this section is organized as follows: we consider the case of week competition in Subsection 2.1. In Subsection 2.2, we will give some results on the case of strong competition.

2.1. **The case of weak competition.** In this subsection, we study the case of week competition, i.e., the case of $0 < a_1 < 1$ and $0 < a_2 < 1$ in (6), under the assumption of $0 < c < c_1^*(\infty)$.

Let $\hat{c}_i(x) = \sqrt{1-a_i}c_i^*(x)$ for $i = 1,2$, and $\hat{c}^*(x) = \min\{\hat{c}_i^*(x)\,|i = 1,2\}$ and

$$\hat{\mu}^*(x) = \min\left\{\sqrt{1-a_1}\mu_2^*(x), \sqrt{1-a_2}\mu_1^*(x)\right\}.$$

Defining

$$\psi(\mu) = 2\sqrt{d_1 d_2}\,\mu,$$

it is easily seen that $\psi(\hat{\mu}^*(\infty)) = \hat{c}^*(\infty)$. Define $H_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}$, $i = 1,2$, by

$$\begin{aligned} &H_1(t,x,v) = v_1[r(x-ct) - v_1 - a_1 r(\infty) + a_1 v_2],\\ &H_2(t,x,v) = [r(\infty) - v_2][r(\infty) - v_2 - r(x-ct) + a_2 v_1], \end{aligned} \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2,$$

where $v = (v_1, v_2)$. Then the translation

$$\begin{cases} v_1(t,x) = u_1(t,x)\\ v_2(t,x) = r(\infty) - u_2(t,x) \end{cases} \tag{10}$$

transfers (6) to the following system

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} = d_1\dfrac{\partial^2 v_1(t,x)}{\partial x^2} + H_1(t,x,v(t,x)),\ t>0,\ x\in\mathbb{R},\\ \dfrac{\partial v_2(t,x)}{\partial t} = d_2\dfrac{\partial^2 v_2(t,x)}{\partial x^2} + H_2(t,x,v(t,x)),\ t>0,\ x\in\mathbb{R},\\ v(0,x) = \theta(x),\ x\in\mathbb{R}, \end{cases} \tag{11}$$

where $\theta(x) = (\theta_1(x), \theta_2(x))$ satisfies $\theta_1(x) = \phi_1(x)$ and $\theta_2(x) = r(\infty) - \phi_2(x)$. It is easy to see that (11) is *quasi-monotone* for $v = (v_1, v_2) \in [0, r(\infty)) \times [0, r(\infty)$ (see Definition 2.1 in [36]). Clearly, the initial value problem (11) with $\vec{0} \le \psi(x) \le \vec{r}(\infty)$ has a unique classical solution $v(t,x)$ with $\vec{0} \le v(t,x) \le \vec{r}(\infty)$.

For any finite $T > 0$, we consider (11) in the finite time interval $(0, T)$:

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} = d_1 \dfrac{\partial^2 v_1(t,x)}{\partial x^2} + H_1(t,x,v(t,x)),\ (t,x) \in (0,T) \times \mathbb{R}, \\ \dfrac{\partial v_2(t,x)}{\partial t} = d_2 \dfrac{\partial^2 v_2(t,x)}{\partial x^2} + H_2(t,x,v(t,x)),\ (t,x) \in (0,T) \times \mathbb{R}, \\ v(0,x) = \theta(x),\ x \in \mathbb{R}. \end{cases} \tag{12}$$

We need a notion of weak upper/lower solutions adopted from [27].

Definition 2.4. We call a vector function $v \equiv (v_1, v_2)$ a continuous weak upper (lower) solution of (12) if v is continuous on $[0,T] \times \mathbb{R}$, $v(0,x) \ge (\le) \theta(x)$ and

$$\frac{\partial v_i(t,x)}{\partial t} \ge (\le) d_i \frac{\partial^2 v_i(t,x)}{\partial x^2} + H_i(t,x,v(t,x))$$

in the distributional sense, i.e., for any $\eta_i \in C^{1,2}([0,T] \times \mathbb{R})$ with $\eta_i \ge 0$ and $\mathrm{supp}\eta_i(t,\cdot) \Subset \mathbb{R}$ (meaning that $\mathrm{supp}\eta_i(t,\cdot)$ is bounded interval in $\mathbb{R}$) for all $t \in [0,T]$, there holds

$$\begin{aligned} &\int_{\mathbb{R}} v_i(t,x)\eta_i(t,x)dx \Big|_{t=0}^{t=T_1} \\ &\quad \ge (\le) \int_0^{T_1} \int_{\mathbb{R}} \left[v_i(s,x) \left(d_i \eta_{i,xx} + \eta_{i,t} \right)(s,x) + \eta_i(s,x) H_i(s,x,v(s,x)) \right] dx ds, \end{aligned} \tag{13}$$

for $T_1 \in [0,T]$, where $\eta_{i,xx}(s,x) = \frac{\partial^2 \eta_i(t,x)}{\partial x^2}\big|_{(t,x)=(s,x)}$ and $\eta_{i,t}(s,x) = \frac{\partial \eta_i(t,x)}{\partial t}\big|_{(t,x)=(s,x)}$, $i = 1,2$.

This definition is a slight variation/modification of Definition 1.1 in [46]. Continuous weak upper and lower solutions were used in [1, 27, 46] in studying reaction-diffusion systems.

Lemma 2.5. *Assume that (12) has a weak upper solution $\widetilde{v} \equiv (\widetilde{v_1}, \widetilde{v_2})$ and weak lower $\widehat{v} \equiv (\widehat{v_1}, \widehat{v_2})$ that are continuous and ordered with $\widehat{v} \le \widetilde{v}$ on $[0,T] \times \mathbb{R}$. Then (12) has a classical solution $v \equiv (v_1, v_2)$ satisfying $\widehat{v} \le v \le \widetilde{v}$ on $[0,T] \times \mathbb{R}$.*

The proof is similar to that of Lemma 1.2 in [46] with some minor modifications, and is given in the appendix for readers' convenience.

Let $\delta_1 = \sqrt{d_2/d_1}$ and $\delta_2 = \sqrt{d_1/d_2}$. Then $\delta_2 < 1 < \delta_1$(as we have assumed $d_1 < d_2$). For fixed $\gamma > 0$, consider the following function of x on the interval $[0, \pi/\gamma]$ parameterized by $\mu > 0$:

$$\varphi_i(\mu; x) = \begin{cases} e^{-\delta_i \mu x} \sin \gamma x, & \text{if } 0 \le x \le \pi/\gamma, \\ 0, & \text{elsewhere}, \end{cases} \quad i = 1,2. \tag{14}$$

Such functions were used in [46, 47] in studying reaction-diffusion systems, in addition to [27]. Obviously, $\varphi_1(\mu; x)$ and $\varphi_2(\mu; x)$ are continuous in x and their second order derivatives in x exist and are continuous at $x \ne 0, \pi/\gamma$. The maximum of $\varphi_i(\mu; x)$ occurs at $\sigma_i(\mu) = \gamma^{-1} \tan^{-1}(\delta_i^{-1} \mu^{-1} \gamma)$ and $\sigma_i(\mu)$ is strictly decreasing in

μ, $i = 1, 2$. Moreover, $\varphi_1(\mu; x) \leq \varphi_2(\mu; x)$ and $b^*\varphi_1(\mu; x) \geq \varphi_2(\mu; x)$ for all $x \in \mathbb{R}$, where $b^* = e^{(\delta_1-\delta_2)\mu\pi/\gamma}$.

The following Lemma is a generalization of Lemma 2.3 in [27], and will play an important role in establishing the persistence of (11).

Lemma 2.6. *Assume that $\hat{c}^*(\infty) > c$. For any ϵ satisfying $0 < \epsilon < \frac{\hat{c}^*(\infty)-c}{3}$, let ℓ be the number such that $\hat{c}^*(\ell) = \hat{c}^*(\infty) - a^*\epsilon$, where $a^* = \min\{1 - a_1, 1 - a_2\}$. Let $0 < \mu_1 < \mu_2 < \mu^*(\ell)$ be such that $\psi(\mu_1) = c + \epsilon$ and $\psi(\mu_2) = \hat{c}^*(\infty) - 2\epsilon$. Let $v(t, x, \theta)$ be the solution of (11) with $\vec{0} \leq \theta(x) \leq \vec{r}(\infty)$. Then for any $\mu \in [\mu_1, \mu_2]$ and sufficiently small $\beta > 0$ and $\gamma > 0$, there holds*

$$\widehat{v}(\mu; t, x) \leq v(t, x, \theta) \leq \widetilde{v}(\mu; t, x), \quad \forall t > 0, \ x \in \mathbb{R}, \tag{15}$$

provided that $\widehat{v}(\mu; 0, x) \leq \theta(x) \leq \widetilde{v}(\mu; 0, x)$ for $x \in \mathbb{R}$, where

$$\begin{aligned}
\widetilde{v_1}(\mu; t, x) &= r(\infty) - \beta\varphi_1(\mu; x - \ell - \psi(\mu)t),\\
\widetilde{v_2}(\mu; t, x) &= r(\infty) - \frac{2}{a_1}\beta\varphi_2(\mu; x - \ell - \psi(\mu)t),\\
\widehat{v_1}(\mu; t, x) &= \frac{2}{a_2}b^*\beta\varphi_1(\mu; x - \ell - \psi(\mu)t),\\
\widehat{v_2}(\mu; t, x) &= \beta\varphi_2(\mu; x - \ell - \psi(\mu)t)
\end{aligned}$$

with φ_1 and φ_2 given by (14).

Proof. For any $T > 0$, let $T_1 \in [0, T]$ and $\eta_i \in C^{1,2}([0, T] \times \mathbb{R})$ with $\eta_i \geq 0$ and $\mathrm{supp}\eta_i(t, \cdot) \Subset \mathbb{R}$ for all $t \in [0, T]$, $i = 1, 2$. Then by (14), we have

$$\begin{aligned}
&\int_0^{T_1}\int_{\mathbb{R}} \varphi_i(\mu; x - \ell - \psi(\mu)s)\eta_{i,xx}(s, x)dxds\\
&= \int_0^{T_1}\int_{\Omega(s)} \eta_i(s, x)\varphi_{i,xx}(\mu; x - \ell - \psi(\mu)s)dxds\\
&+ \gamma\int_0^{T_1}\Big[\eta_i(s, \ell + \psi(\mu)s + \pi/\gamma)e^{-\pi\delta_i\mu/\gamma} + \eta_i(s, \ell + \psi(\mu)s)\Big]ds
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
&\int_0^{T_1}\int_{\mathbb{R}} \varphi_i(\mu; x - \ell - \psi(\mu)s)\eta_{i,t}(s, x)dxds\\
&= \int_{\mathbb{R}} \varphi_i(\mu; x - \ell - \psi(\mu)t)\eta_i(t, x)dx\,\Big|_{t=0}^{t=T_1}\\
&- \int_0^{T_1}\int_{\mathbb{R}} \eta_i(s, x)\varphi_{i,t}(\mu; x - \ell - \psi(\mu)s)dxds,
\end{aligned}\tag{17}$$

where $\Omega(s) = \{x\,|x \in \mathbb{R}, \ell + \psi(\mu)s \leq x \leq \ell + \psi(\mu)s + \pi/\gamma\}$ and for $i = 1, 2$,

$$\begin{aligned}
\varphi_{i,xx}(\mu; x - \ell - \psi(\mu)s) &= \frac{\partial^2\varphi_i(\mu; x - \ell - \psi(\mu)t)}{\partial x^2}\,\Big|_{(t,x)=(s,x)},\\
\varphi_{i,t}(\mu; x - \ell - \psi(\mu)s) &= \frac{\partial\varphi_i(\mu; x - \ell - \psi(\mu)t)}{\partial t}\,\Big|_{(t,x)=(s,x)}.
\end{aligned}$$

Direct calculations show that

$$(\varphi_{1,t} - d_1\varphi_{1,xx})(\mu; x - \ell - \psi(\mu)t) = (d_2\mu^2 + d_1\gamma^2)\varphi_1(\mu; x - \ell - \psi(\mu)t), \tag{18}$$

$$(\varphi_{2,t} - d_2\varphi_{2,xx})(\mu; x - \ell - \psi(\mu)t) = (d_1\mu^2 + d_2\gamma^2)\varphi_2(\mu; x - \ell - \psi(\mu)t) \tag{19}$$

for all $\mu \in [\mu_1, \mu_2]$, $x \neq \ell + \psi(\mu)t$ and $x \neq \ell + \psi(\mu)t + \pi/\gamma$. Thus, for any $\mu \in [\mu_1, \mu_2]$, we have

$$\begin{aligned} r(\ell) - a_1 r(\infty) - d_2\mu^2 &= \frac{(1-a_1)\hat{c}^{*2}(\infty) - 2\epsilon a^* \hat{c}^*(\infty) + a^{*2}\epsilon^2}{4\min\{d_1(1-a_1), d_2(1-d_2)\}} - d_2\mu^2 \\ &\geq \frac{(1-a_1)\hat{c}^{*2}(\infty) - 2\epsilon a^* \hat{c}^*(\infty) + a^{*2}\epsilon^2}{4d_1(1-a_1)} - \frac{\hat{c}^{*2}(\infty) - 4\epsilon \hat{c}^*(\infty) + 4\epsilon^2}{4d_1} \\ &\geq \frac{\epsilon[\hat{c}^*(\infty) - 2\epsilon]}{2d_1} > \frac{\epsilon[c+\epsilon]}{2d_1} \end{aligned} \tag{20}$$

and

$$\begin{aligned} r(\ell) - a_2 r(\infty) - d_1\mu^2 &= \frac{(1-a_2)\hat{c}^{*2}(\infty) - 2\epsilon a^* \hat{c}^*(\infty) + a^{*2}\epsilon^2}{4\min\{d_1(1-a_1), d_2(1-d_2)\}} - d_1\mu^2 \\ &\geq \frac{(1-a_2)\hat{c}^{*2}(\infty) - 2\epsilon a^* \hat{c}^*(\infty) + a^{*2}\epsilon^2}{4d_2(1-a_2)} - \frac{\hat{c}^{*2}(\infty) - 4\epsilon \hat{c}^*(\infty) + 4\epsilon^2}{4d_2} \\ &\geq \frac{\epsilon[\hat{c}^*(\infty) - 2\epsilon]}{2d_2} > \frac{\epsilon[c+\epsilon]}{2d_2}. \end{aligned} \tag{21}$$

It follows from (16)-(21) that for any $\mu \in [\mu_1, \mu_2]$ and for sufficiently small $\beta > 0$ and $\gamma > 0$, $\widehat{v}(\mu; t, x)$ satisfies

$$\begin{aligned} &\int_0^{T_1}\int_{\mathbb{R}} [\widehat{v}_1(\mu; s, x)(d_1\eta_{1,xx} + \eta_{1,t})(s,x) + \eta_1(s,x)H_1(s,x,\widehat{v}(\mu; s,x))]\,dxds \\ &- \int_{\mathbb{R}} \widehat{v}_1(\mu; t, x)\eta_1(t,x)dx \Big|_{t=0}^{t=T_1} \\ &= \int_0^{T_1}\int_{\Omega(s)} \left[\frac{2}{a_2}b^*\beta(d_1\varphi_{1,xx} - \varphi_{1,t})(\mu; x-\ell-\psi(\mu)s) + H_1(s,x,\widehat{v}(\mu; s,x))\right] \\ &\times \eta_1(s,x)dxds - \frac{2}{a_2}b^*\beta\int_0^{T_1}\int_{\mathbb{R}/\Omega(s)} \eta_1(s,x)\varphi_{1,t}(\mu; x-\ell-\psi(\mu)s)dxds \\ &+ \frac{2}{a_2}b^*\beta\gamma\int_0^{T_1}\left[\eta_1(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_1\mu/\gamma} + \eta_1(s,\ell+\psi(\mu)s)\right]ds \\ &= \int_0^{T_1}\int_{\Omega(s)} \left[-(d_2\mu^2 + d_1\gamma^2)\widehat{v}_1(\mu; s,x) + H_1(s,x,\widehat{v}(\mu; s,x))\right]\eta_1(s,x)dxds \\ &+ \frac{2}{a_2}b^*\beta\gamma\int_0^{T_1}\left[\eta_1(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_1\mu/\gamma} + \eta_1(s,\ell+\psi(\mu)s)\right]ds \\ &\geq \int_0^{T_1}\int_{\Omega(s)} \left[r(\ell) - a_1 r(\infty) - d_2\mu^2 - d_1\gamma^2 - \widehat{v}_1(\mu; s,x) + a_1\widehat{v}_2(\mu; s,x)\right] \\ &\times \widehat{v}_1(\mu; s,x)\eta_1(s,x)dxds \\ &+ \frac{2}{a_2}b^*\beta\gamma\int_0^{T_1}\left[\eta_1(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_1\mu/\gamma} + \eta_1(s,\ell+\psi(\mu)s)\right]ds \\ &\geq \int_0^{T_1}\int_{\Omega(s)} \left[\frac{\epsilon(c+\epsilon)}{2d_1} - d_1\gamma^2 - \widehat{v}_1(\mu; s,x) + a_1\widehat{v}_2(\mu; s,x)\right]\widehat{v}_1(\mu; s,x)\eta_1(s,x)dxds \\ &+ \frac{2}{a_2}b^*\beta\gamma\int_0^{T_1}\left[\eta_1(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_1\mu/\gamma} + \eta_1(s,\ell+\psi(\mu)s)\right]ds \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned}
&\int_0^{T_1}\int_{\mathbb{R}} [\widehat{v}_2(\mu;s,x)(d_2\eta_{2,xx}+\eta_{2,t})(s,x)+\eta_2(s,x)H_2(s,x,\widehat{v}(\mu;s,x))]\,dxds\\
&-\int_{\mathbb{R}}\widehat{v}_2(\mu;t,x)\eta_2(t,x)dx\Big|_{t=0}^{t=T_1}\\
&=\int_0^{T_1}\int_{\Omega(s)}[\beta(d_2\varphi_{2,xx}-\varphi_{2,t})(\mu;x-\ell-\psi(\mu)s)+H_2(s,x,\widehat{v}(\mu;s,x))]\\
&\times\eta_2(s,x)dxds-\beta\int_0^{T_1}\int_{\mathbb{R}\setminus\Omega(s)}\eta_2(s,x)\varphi_{2,t}(\mu;x-\ell-\psi(\mu)s)dxds\\
&+\beta\gamma\int_0^{T_1}\Big[\eta_2(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_2\mu/\gamma}+\eta_2(s,\ell+\psi(\mu)s)\Big]\,ds\\
&=\int_0^{T_1}\int_{\Omega(s)}\left[-(d_1\mu^2+d_2\gamma^2)\widehat{v}_2(\mu;s,x)+H_2(s,x,\widehat{v}(\mu;s,x))\right]\eta_2(s,x)dxds\\
&+\beta\gamma\int_0^{T_1}\Big[\eta_2(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_2\mu/\gamma}+\eta_2(s,\ell+\psi(\mu)s)\Big]\,ds\\
&\geq\int_0^{T_1}\int_{\Omega(s)}\left[r(\infty)-d_1\mu^2-d_2\gamma^2-\widehat{v}_2(\mu;s,x)\right]\widehat{v}_2(\mu;s,x)\eta_2(s,x)dxds\\
&+\beta\gamma\int_0^{T_1}\Big[\eta_2(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_2\mu/\gamma}+\eta_2(s,\ell+\psi(\mu)s)\Big]\,ds\\
&\geq\int_0^{T_1}\int_{\Omega(s)}\left[a_2r(\infty)+\frac{\epsilon(c+\epsilon)}{2d_2}-d_2\gamma^2-\widehat{v}_2(\mu;s,x)\right]\widehat{v}_2(\mu;s,x)\eta_2(s,x)dxds\\
&+\beta\gamma\int_0^{T_1}\Big[\eta_2(s,\ell+\psi(\mu)s+\pi/\gamma)e^{-\pi\delta_2\mu/\gamma}+\eta_2(s,\ell+\psi(\mu)s)\Big]\,ds\\
&\geq 0,
\end{aligned}$$

where $\mathbb{R}\setminus\Omega(s)=\{x|x\in\mathbb{R},x>\ell+\psi(\mu)s+\pi/\gamma\}\bigcup\{x|x\in\mathbb{R},x<\ell+\psi(\mu)s\}$. Here, we have used the inequality $H_2(s,x,\widehat{v}(\mu;s,x))\geq\widehat{v}_2[r(\infty)-\widehat{v}_2]$ which is a consequence of the relation $b^*\varphi_1(\mu;x)\geq\varphi_2(\mu;x)$ together with the definition of $\widehat{v}=(\widehat{v}_1,\widehat{v}_2)$ in Lemma 2.6. It follows from Definition 2.4 that for any $\mu\in[\mu_1,\mu_2]$ and for sufficiently small $\beta>0$ and $\gamma>0$, $\widehat{v}(\mu;t,x)$ is a continuous weak lower solution of (12). By Lemma 2.5, the first half of (15) holds on $[0,T]\times\mathbb{R}$. Because $T>0$ is arbitrary, we have actually shown that the first half of (15) holds on $\mathbb{R}_+\times\mathbb{R}$.

To prove the second inequality of (15), we transfer (11) to the following quasi-monotone system

$$\begin{cases}\dfrac{\partial v_1^*(t,x)}{\partial t}=d_1\dfrac{\partial^2 v_1^*(t,x)}{\partial x^2}+H_1^*(t,x,v^*(t,x)),\ t>0,\ x\in\mathbb{R},\\ \dfrac{\partial v_2^*(t,x)}{\partial t}=d_2\dfrac{\partial^2 v_2^*(t,x)}{\partial x^2}+H_2^*(t,x,v^*(t,x)),\ t>0,\ x\in\mathbb{R},\\ v^*(0,x)=\phi^*(x),\ x\in\mathbb{R},\end{cases}\tag{22}$$

by the translation

$$\begin{cases}v_1^*(t,x)=r(\infty)-v_1(t,x),\\ v_2^*(t,x)=r(\infty)-v_2(t,x),\end{cases}\tag{23}$$

where $v^*(t,x) = (v_1^*(t,x), v_2^*(t,x))$, $\phi^*(x) = (\phi_1^*(x), \phi_2^*(x))$ with $\phi_1^*(x) = r(\infty) - \theta_1(x)$ and $\phi_2^*(x) = r(\infty) - \theta_2(x)$, and

$$H_1^*(t,x,v) = [r(\infty) - v_1^*][r(\infty) - r(x-ct) - v_1^* + a_1 v_2^*],$$
$$H_2^*(t,x,v) = v_2^*[r(x-ct) - a_2 r(\infty) - v_2^* + a_2 v_1^*].$$

Clearly, the initial value problem (22) with $\vec{0} \le \phi^*(x) \le \vec{r}(\infty)$ has a unique classical solution $v^*(t,x,\phi^*)$ satisfying $\vec{0} \le v^*(t,x,\phi^*) \le \vec{r}(\infty)$ and $v^*(t,x,\phi^*) = r(\infty) - v(t,x,\theta)$. Let

$$\begin{cases} \hat{v}_1^*(\mu;t,x) = r(\infty) - \tilde{v}_1(\mu;t,x), \\ \hat{v}_2^*(\mu;t,x) = r(\infty) - \tilde{v}_2(\mu;t,x). \end{cases} \tag{24}$$

Then

$$\begin{cases} \hat{v}_1^*(\mu;t,x) = \beta\varphi_1(\mu; x-\ell-\psi(\mu)t), \\ \hat{v}_2^*(\mu;t,x) = \dfrac{2}{a_1}\beta\varphi_2(\mu; x-\ell-\psi(\mu)t). \end{cases} \tag{25}$$

Repeating the proof of the first half of (15) with v replaced by v^*, $\hat{v}$ by $\hat{v}^*$ and θ by ϕ^*, we have

$$\hat{v}^*(\mu;t,x) \le v^*(t,x,\phi^*), \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

which leads to the second half of (15), completing the proof. □

The following theorem shows that if $\hat{c}^*(\infty) > c$, two competing species can both persist in space and spread to the right at the asymptotic spreading speed *is larger* than $\hat{c}^*(\infty)$.

Theorem 2.7. *Assume that (A) holds,* $0 < a_j < 1$ *for* $j = 1,2$, *and* $0 < c < \hat{c}^*(\infty)$. *Suppose* $\phi_i(x) > 0$ *on a closed interval,* $i = 1,2$. *Let* $u(t,x,\phi) = (u_1(t,x,\phi), u_2(t,x,\phi)$ *be the solution to* (6) *with* $\vec{0} \le \phi(x) \le \vec{r}(\infty)$. *Then, for any* ε *with* $0 < \varepsilon < (\hat{c}^*(\infty) - c)/2$,

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}\left|\frac{1-a_1}{1-a_1a_2}r(\infty) - u_1(t,x,\phi)\right|\right] = 0$$

and

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}\left|\frac{1-a_2}{1-a_1a_2}r(\infty) - u_2(t,x,\phi)\right|\right] = 0.$$

Proof. For any $\delta > 0$ satisfying $0 < \delta < \min\left\{\frac{1}{3}, r(\infty), \frac{\hat{c}^*(\infty)-c}{3}\right\}$, let ℓ be a positive number such that

$$\hat{c}^*(\ell) = \hat{c}^*(\infty) - a^*\delta,$$

where $a^* = \min\{1-a_1, 1-a_2\}$. Choose $0 < \mu_1 < \mu_2 < \mu^*(\ell)$ such that $\psi(\mu_1) = c+\delta$ and $\psi(\mu_2) = \hat{c}^*(\infty) - 2\delta$. Let $v(t,x,\theta)$ be the solution of (11) with $\theta_1(x) = \phi_1(x)$ and $\theta_2(x) = r(\infty) - \phi_2(x)$. By Lemma 2.6, for any $\mu \in [\mu_1, \mu_2]$ and sufficiently small $\beta > 0$ and $\gamma > 0$,

$$\underline{v}(\mu;t,x) \le v(t,x,\theta) \le \overline{v}(\mu;t,x), \quad \forall t > 0,\ x \in \mathbb{R} \tag{26}$$

provided that $\underline{v}(\mu;0,x) \le \theta(x) \le \overline{v}(\mu;0,x)$ for all $x \in \mathbb{R}$, where

$$\begin{cases} \underline{v}(\mu;t,x) = (\underline{v}_1(\mu;t,x), \underline{v}_2(\mu;t,x)), \\ \overline{v}(\mu;t,x) = (\overline{v}_1(\mu;t,x), \overline{v}_2(\mu;t,x)) \end{cases} \tag{27}$$

and

$$\begin{cases} \underline{v}_i(\mu;t,x) = \dfrac{\underline{\alpha}_i}{\varphi_i(\mu;\sigma_i(\mu))}\varphi_i(\mu;x-\ell-\psi(\mu)t),\ i=1,2, \\ \overline{v}_i(\mu;t,x) = r(\infty) - \dfrac{\overline{\alpha}_i}{\varphi_i(\mu;\sigma_i(\mu))}\varphi_i(\mu;x-\ell-\psi(\mu)t),\ i=1,2, \end{cases} \tag{28}$$

where $\underline{\alpha}_1 = \frac{2}{a_2}b^*\beta$, $\underline{\alpha}_2 = \overline{\alpha}_1 = \beta$ and $\overline{\alpha}_2 = \frac{2}{a_1}\beta$.

Since $\phi_1(x) > 0$ on a closed interval, we have that $\theta_1(x) > 0$ on the closed interval. Therefore, $v_i(t,x,\theta) > 0$ for all $x \in \mathbb{R}$ and $t > 0$, $i = 1,2$. Choose $0 < t_0 < \min\left\{\frac{\sigma_1(\mu_1)}{\psi(\mu_1)}, \frac{\sigma_2(\mu_1)}{\psi(\mu_1)}\right\}$, and choose β and γ sufficiently small such that $v_i(t_0,x,\theta) > \underline{\alpha}_i$ for all $x \in [\ell, \ell+4\pi/\gamma]$, $i=1,2$. Define

$$\underline{w}_i(0,x) = \begin{cases} \dfrac{\underline{\alpha}_i}{\varphi_i(\mu_1;\sigma_i(\mu_1))}\varphi_i(\mu_1;x-\ell) \ \text{ if } \ell \le x \le \ell+\sigma_i(\mu_1); \\ \underline{\alpha}_i \ \text{ if } \ell+\sigma_i(\mu_1) \le x \le \ell+3\pi/\gamma+\sigma_i(\mu_2); \\ \dfrac{\underline{\alpha}_i}{\varphi_i(\mu_2;\sigma_i(\mu_2))}\varphi_i(\mu_2;x-\ell-3\pi/\gamma) \\ \qquad \text{if } \ell+3\pi/\gamma+\sigma_i(\mu_2) \le x \le \ell+4\pi/\gamma; \\ 0 \ \text{ elsewhere} \end{cases}$$

for $i=1,2$. It is easily seen that

$$\underline{w}_i(0,x) \ge \frac{\underline{\alpha}_i}{\varphi_i(\mu_1;\sigma_i(\mu_1))}\varphi_i(\mu_1;x-\ell-s),\ \forall s \in [0,2\pi/\gamma]$$

and

$$\underline{w}_i(0,x) \ge \frac{\underline{\alpha}_i}{\varphi_i(\mu_2;\sigma_i(\mu_2))}\varphi_i(\mu_2;x-\ell-3\pi/\gamma+s),\ \forall s \in [0,2\pi/\gamma]$$

for $i=1,2$. Since $v_i(t_0,x,\theta) > \underline{\alpha}_i$ for all $x \in [\ell,\ell+4\pi/\gamma]$, Lemma 2.6 implies that

$$v_i(t,x,\theta) \ge \frac{\underline{\alpha}_i}{\varphi_i(\mu_1;\sigma_i(\mu_1))}\varphi_i(\mu_1;x-\ell-\psi(\mu_1)(t-t_0)-s) \tag{29}$$

and

$$v_i(t,x,\theta) \ge \frac{\underline{\alpha}_i}{\varphi_i(\mu_2;\sigma_i(\mu_2))}\varphi_i(\mu_2;x-\ell-3\pi/\gamma-\psi(\mu_2)(t-t_0)+s) \tag{30}$$

for all $t \ge t_0$ and $s \in [0,2\pi/\gamma]$, $i=1,2$. By (29) and (30), for $t \ge t_0$ and $i=1,2$, we have

$$v_i(t,x,\theta) \ge \begin{cases} \dfrac{\underline{\alpha}_i}{\varphi_i(\mu_1;\sigma_i(\mu_1))}\varphi_i(\mu_1;x-\ell-\psi(\mu_1)(t-t_0)) \\ \qquad \text{if } \ell+\psi(\mu_1)(t-t_0) \le x \le \ell+\sigma_i(\mu_1)+\psi(\mu_1)(t-t_0); \\ \underline{\alpha}_i \quad \text{if } \ell+\sigma_i(\mu_1)+\psi(\mu_1)(t-t_0) \le x \le \ell+\sigma_i(\mu_1)+2\pi/\gamma+\psi(\mu_1)(t-t_0); \\ 0 \qquad \text{elsewhere} \end{cases} \tag{31}$$

and

$$v_i(t,x,\theta) \ge \begin{cases} \underline{\alpha}_i \quad \text{if } \ell+\sigma_i(\mu_2)+\pi/\gamma+\psi(\mu_2)(t-t_0) \le x \\ \qquad\qquad \le \ell+\sigma_i(\mu_2)+3\pi/\gamma+\psi(\mu_2)(t-t_0); \\ \dfrac{\underline{\alpha}_i}{\varphi_i(\mu_2;\sigma_i(\mu_2))}\varphi_i(\mu_2;x-\ell-3\pi/\gamma-\psi(\mu_2)(t-t_0)) \\ \qquad \text{if } \ell+\sigma_i(\mu_2)+3\pi/\gamma+\psi(\mu_2)(t-t_0) \le x \le \ell+4\pi/\gamma+\psi(\mu_2)(t-t_0); \\ 0 \qquad \text{elsewhere.} \end{cases} \tag{32}$$

Let

$$h_i = \frac{\pi/\gamma + \sigma_i(\mu_1) - \sigma_i(\mu_2)}{\psi(\mu_2) - \psi(\mu_1)}, \quad i = 1, 2.$$

Since

$$\ell + \sigma_i(\mu_2) + \pi/\gamma + \psi(\mu_2)(t - t_0) \le \ell + \sigma_i(\mu_1) + 2\pi/\gamma + \psi(\mu_1)(t - t_0)$$

for all $t \in [t_0, t_0 + h_i]$, (31) and (32) imply that

$$v_i(t, x, \theta) \ge \underline{w}_i(t - t_0, x) \tag{33}$$

for all $t \in [t_0, t_0 + h_i]$, $i = 1, 2$, where

$$\underline{w}_i(t - t_0, x) = \begin{cases} \dfrac{\underline{\alpha}_i}{\varphi_i(\mu_1; \sigma_i(\mu_1))} \varphi_i(\mu_1; x - \ell - \psi(\mu_1)(t - t_0)) \\ \quad \text{if } \ell + \psi(\mu_1)(t - t_0) \le x \le \ell + \sigma_i(\mu_1) + \psi(\mu_1)(t - t_0); \\ \underline{\alpha}_i \quad \text{if } \ell + \sigma_i(\mu_1) + \psi(\mu_1)(t - t_0) \le x \le \ell + \sigma_i(\mu_2) + 3\pi/\gamma + \psi(\mu_2)(t - t_0); \\ \dfrac{\underline{\alpha}_i}{\varphi_i(\mu_2; \sigma_i(\mu_2))} \varphi_i(\mu_2; x - \ell - 3\pi/\gamma - \psi(\mu_2)(t - t_0)) \\ \quad \text{if } \ell + \sigma_i(\mu_2) + 3\pi/\gamma + \psi(\mu_2)(t - t_0) \le x \le \ell + 4\pi/\gamma + \psi(\mu_2)(t - t_0); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{34}$$

By the same arguments as in the proof of Theorem 2.2 of [27], we can see that (33) holds for all $t \ge t_0$.

For the chosen $\delta > 0$ and $\beta > 0$, there exists $L > 0$ such that

$$\int_{-L}^{L} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \ge 1 - \beta\delta.$$

Therefore, for any $s > 0$,

$$\int_{-L\sqrt{4d_i s}}^{L\sqrt{4d_i s}} \frac{1}{\sqrt{4\pi d_i s}} e^{-\frac{x^2}{4d_i s}} dx \ge 1 - \beta\delta, \ i = 1, 2.$$

Let $t_1 > t_0$ be sufficiently large. Then, for $t > t_1$, the solution $v(t, x) \equiv v(t, x, \theta)$ of (11) satisfies the integral equation

$$v_i(t, x) = \int_{\mathbb{R}} k_i(t - t_1, x - y) v_i(t_1, y) dy + \int_{t_1}^{t} \int_{\mathbb{R}} k_i(t - s, x - y) H_{i,\rho}(s, y, v(s, y)) dy ds, \tag{35}$$

where $H_{i,\rho}(t, x, v) = \rho v_i + H_i(t, x, v)$, $\rho = 3\rho - r(-\infty)$ and

$$k_i(t, x) = \frac{1}{\sqrt{4\pi d_i t}} e^{-\rho t - \frac{x^2}{4d_i t}}, \ i = 1, 2. \tag{36}$$

By (33) and (35), we have

$$\begin{aligned} v_i(t, x) \ge & \int_{\mathbb{R}} k_i(t - t_1, x - y) \underline{w}_i(t_1 - t_0, y) dy \\ & + \int_{t_1}^{t} \int_{\mathbb{R}} k_i(t - s, x - y) H_{i,\rho}(s, y, \underline{w}(s - t_0, y)) dy ds \end{aligned} \tag{37}$$

for all $t > t_1$, $i = 1, 2$, where $\underline{w}(t, x) = (\underline{w}_1(t, x), \underline{w}_2(t, x))$. For $t > t_1$ and x, y satisfying

$$\begin{aligned} & \ell + \sigma_i(\mu_1) + \psi(\mu_1)(t_1 - t_0) + L\sqrt{4d_i(t - t_1)} \\ & \quad \le x \le \ell + \sigma_i(\mu_2) + \psi(\mu_2)(t_1 - t_0) + 3\pi/\gamma - L\sqrt{4d_i(t - t_1)} \end{aligned} \tag{38}$$

and

$$-L\sqrt{4d_i(t-t_1)} \le y \le L\sqrt{4d_i(t-t_1)}, \tag{39}$$

we have that

$$\ell+\sigma_i(\mu_1)+\psi(\mu_1)(t_1-t_0) \le x-y \le \ell+\sigma_i(\mu_2)+\psi(\mu_2)(t_1-t_0)+3\pi/\gamma, \quad i=1,2. \tag{40}$$

Therefore, by (34) and (40), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}} k_i(t-t_1, x-y)\underline{w}_i(t_1-t_0, y)dy \\
=&\int_{\mathbb{R}} k_i(t-t_1, y)\underline{w}_i(t_1-t_0, x-y)dy \\
\ge& e^{-\rho(t-t_1)} \int_{-L\sqrt{4d_i(t-t_1)}}^{L\sqrt{4d_i(t-t_1)}} \frac{1}{\sqrt{4\pi d_i(t-t_1)}} e^{-\frac{y^2}{4d_i(t-t_1)}} \underline{w}_i(t_1-t_0, x-y)dy \\
=&\underline{\alpha}_i e^{-\rho(t-t_1)} \int_{-L\sqrt{4d_i(t-t_1)}}^{L\sqrt{4d_i(t-t_1)}} \frac{1}{\sqrt{4\pi d_i(t-t_1)}} e^{-\frac{y^2}{4d_i(t-t_1)}} dy \\
\ge&(1-\beta\delta)\underline{\alpha}_i e^{-\rho(t-t_1)}, \quad i=1,2
\end{aligned} \tag{41}$$

for all x satisfying (38). Let $\widetilde{\sigma}(\mu_1) = \max\{\sigma_1(\mu_1), \sigma_2(\mu_1)\}$ and $\widehat{\sigma}(\mu_2) = \min\{\sigma_1(\mu_2), \sigma_2(\mu_2)\}$. For $t > t_1$, x and y satisfying

$$\begin{aligned}
&\ell + \widetilde{\sigma}(\mu_1) + \psi(\mu_1)(s-t_0) + L\sqrt{4d_i(t-s)} \\
&\quad \le x \le \ell + \widehat{\sigma}(\mu_2) + \psi(\mu_2)(s-t_0) + 3\pi/\gamma - L\sqrt{4d_i(t-s)}, \quad s \in [t_1, t], \ i=1,2
\end{aligned} \tag{42}$$

and

$$-L\sqrt{4d_i(t-s)} \le y \le L\sqrt{4d_i(t-s)}, \quad s \in [t_1, t], \ i=1,2, \tag{43}$$

we have

$$\ell + \widetilde{\sigma}(\mu_1) + \psi(\mu_1)(s-t_0) \le x-y \le \ell + \widehat{\sigma}(\mu_2) + \psi(\mu_2)(s-t_0) + 3\pi/\gamma \tag{44}$$

and

$$x-y-cs \ge \ell + \widetilde{\sigma}(\mu_1) + \psi(\mu_1)(s-t_0) - cs = \ell + \widetilde{\sigma}(\mu_1) + \epsilon s - t_0\psi(\mu_1) > \ell \tag{45}$$

for all $s \in [t_1, t]$. Define $\widehat{H}_{i,\rho} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $i=1,2$, by

$$\begin{aligned}
\widehat{H}_{1,\rho}(v) &= v_1(\rho + (1-a_1)r(\infty) - v_1 + a_1 v_2 - \ell^*\epsilon) \\
\widehat{H}_{2,\rho}(v) &= \rho v_2 + (r(\infty) - v_2)(a_2 v_1 - v_2)
\end{aligned}$$

where $v = (v_1, v_2)$, $\ell^* = 2a^* r(\infty)/\hat{c}^*(\infty)$. Then it follows from (34), (44) and (45) that

$$\begin{aligned}
&\int_{t_1}^{t}\int_{\mathbb{R}} k_i(t-s, x-y) H_{i,\rho}(s, y, \underline{w}(s-t_0, y))dyds \\
=&\int_{t_1}^{t}\int_{\mathbb{R}} k_i(t-s, y) H_{i,\rho}(s, x-y, \underline{w}(s-t_0, x-y))dyds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{t_1}^{t} e^{-\rho(t-s)} \int_{-L\sqrt{4d_i(t-s)}}^{L\sqrt{4d_i(t-s)}} \frac{1}{\sqrt{4\pi d_i(t-s)}} e^{-\frac{y^2}{4d_i(t-s)}} H_{i,\rho}(s, x-y, \underline{w}(s-t_0, x-y)) dyds \\
&\geq \widehat{H}_{i,\rho}(\underline{\alpha}_1, \underline{\alpha}_2) \int_{t_1}^{t} e^{-\rho(t-s)} \int_{-L\sqrt{4d_i(t-s)}}^{L\sqrt{4d_i(t-s)}} \frac{1}{\sqrt{4\pi d_i(t-s)}} e^{-\frac{y^2}{4d_i(t-s)}} dyds \\
&\geq (1-\beta\delta)\widehat{H}_{i,\rho}(\underline{\alpha}_1, \underline{\alpha}_2) \int_{t_1}^{t} e^{-\rho(t-s)} ds
\end{aligned}
\tag{46}
$$

for all x satisfying (42), $i = 1, 2$. Here we have used the fact that

$$
r(\ell) = r(\infty) - \frac{a^*[2\hat{c}^*(\infty) - a^*\delta]\delta}{\min\{4d_1(1-a_1), 4d_2(1-a_2)\}} > r(\infty) - \ell^*\delta.
$$

By (37), (41) and (46), we then have

$$
v_i(t,x) \geq \hat{v}_i^{(1)}(t) \tag{47}
$$

for all $t > t_1$ and x satisfying (38) and (42), where

$$
\hat{v}_i^{(1)}(t) = (1-\beta\epsilon)\underline{\alpha}_i e^{-\rho(t-t_1)} + (1-\beta\epsilon)\int_{t_1}^{t} e^{-\rho(t-s)} \widehat{H}_{i,\rho}(\underline{\alpha}_1, \underline{\alpha}_2) ds, \quad i = 1, 2. \tag{48}
$$

It then further follows from (35) and induction that

$$
v_i(t,x) \geq \hat{v}_i^{(n)}(t) \tag{49}
$$

for all $t > t_1$ and x satisfying (38) and

$$
\begin{aligned}
&\ell + \widetilde{\sigma}(\mu_1) + \psi(\mu_1)(s-t_0) + nL\sqrt{4d_i(t-s)} \\
&\qquad \leq x \leq \ell + \widehat{\sigma}(\mu_2) + \psi(\mu_2)(s-t_0) + 3\pi/\gamma - nL\sqrt{4d_i(t-s)}, \quad \forall s \in [t_1, t],
\end{aligned}
\tag{50}
$$

where

$$
\hat{v}_i^{(n)}(t) = (1-\beta\delta)\underline{\alpha}_i e^{-\rho(t-t_1)} + (1-\beta\delta)\int_{t_1}^{t} e^{-\rho(t-s)} \widehat{H}_{i,\rho}(\hat{v}_1^{(n-1)}, \hat{v}_2^{(n-1)}) ds, \tag{51}
$$

for $i = 1, 2$. Direct calculations and induction show that

$$
\hat{v}_i^{(n)}(t) = \hat{a}_i^{(n)} + \hat{b}_i^{(n)}(t) e^{-\rho(t-t_1)}, \tag{52}
$$

where for $i = 1, 2$,

$$
\hat{a}_i^{(n)} = (1-\beta\delta)\widehat{H}_{i,\rho}(\hat{a}_1^{(n-1)}, \hat{a}_2^{(n-1)})/\rho, \tag{53}
$$

$$
\hat{a}_i^{(1)} = (1-\beta\delta)\widehat{H}_{i,\rho}(\underline{\alpha}_1, \underline{\alpha}_2)/\rho, \tag{54}
$$

and $\hat{b}_i^{(n)}(t)$ is a sum of products of polynomials and exponential functions of the form $e^{-j\rho(t-t_1)}$ with j being a non-negative integer. Observe that

$$
\lim_{t\to\infty} \hat{v}_i^{(n)}(t) = \hat{a}_i^{(n)}, \ i = 1, 2. \tag{55}
$$

Therefore, $\hat{a}_i^{(n)} \leq r(\infty)$ for all $n \geq 1$, $i = 1, 2$. Let $\hat{a}_i^{(0)} = \underline{\alpha}_i$, $i = 1, 2$. Then for small δ and β, we obtain

$$
\hat{a}_1^{(1)} - \hat{a}_1^{(0)} = \underline{\alpha}_1 \left[(1-a_1)r(\infty) - \tfrac{\beta\delta}{1-\beta\delta}\rho - \underline{\alpha}_1 + a_1\underline{\alpha}_2 - \ell^*\delta\right] \tfrac{1-\beta\delta}{\rho} > 0 \tag{56}
$$

and

$$
\hat{a}_2^{(1)} - \hat{a}_2^{(0)} = \beta \left[(r(\infty) - \beta)(2b^* - 1) - \tfrac{\delta}{1-\beta\delta}\rho\right] \tfrac{1-\beta\delta}{\rho} > 0. \tag{57}
$$

It follows from (56), (57) and induction that

$$\begin{aligned}\hat{a}_1^{(n+1)} - \hat{a}_1^{(n)} =& \left[\widehat{H}_{1,\rho}(\hat{a}_1^{(n)}, \hat{a}_2^{(n)}) - \widehat{H}_{1,\rho}(\hat{a}_1^{(n-1)}, \hat{a}_2^{(n-1)})\right] \frac{1-\beta\delta}{\rho} \\ =&(\hat{a}_1^{(n)} - \hat{a}_1^{(n-1)}) \left[\rho + (1-a_1)r(\infty) - \ell^*\delta - \hat{a}_1^{(n)} - \hat{a}_1^{(n-1)}\right] \frac{1-\beta\delta}{\rho} \\ &+ a_1(\hat{a}_2^{(n)} - \hat{a}_2^{(n-1)})(\hat{a}_2^{(n)} + \hat{a}_2^{(n-1)})\frac{1-\beta\delta}{\rho} \\ >&0\end{aligned} \tag{58}$$

and

$$\begin{aligned}\hat{a}_2^{(n+1)} - \hat{a}_2^{(n)} =& \left[\widehat{H}_{2,\rho}(\hat{a}_1^{(n)}, \hat{a}_2^{(n)}) - \widehat{H}_{2,\rho}(\hat{a}_1^{(n-1)}, \hat{a}_2^{(n-1)})\right] \frac{1-\beta\delta}{\rho} \\ =&(\hat{a}_2^{(n)} - \hat{a}_2^{(n-1)}) \left[\rho - r(\infty) + \hat{a}_2^{(n)} + \hat{a}_2^{(n-1)} - a_2\hat{a}_1^{(n)}\right] \frac{1-\beta\delta}{\rho} \\ &+ a_2(\hat{a}_1^{(n)} - \hat{a}_1^{(n-1)})(r(\infty) - \hat{a}_2^{(n-1)})\frac{1-\beta\delta}{\rho} \\ >&0\end{aligned} \tag{59}$$

for all $n \geq 1$. Thus, $\left\{\hat{a}_1^{(n)}\right\}_{n=0}^{\infty}$ and $\left\{\hat{a}_2^{(n)}\right\}_{n=0}^{\infty}$ are both increasing and $0 < \hat{a}_i^{(n)} \leq r(\infty)$ for $n \geq 0$, $i = 1, 2$. So,

$$\lim_{n\to\infty} \hat{a}_1^{(n)} \quad \text{and} \quad \lim_{n\to\infty} \hat{a}_1^{(n)}$$

both exist. Let

$$\lim_{n\to\infty} \hat{a}_i^{(n)} = \hat{a}_i^*, \quad i = 1, 2. \tag{60}$$

Then (53) and direct calculations show that

$$\begin{cases} \hat{a}_1^* = a_1\hat{a}_2^* + (1-a_1)r(\infty) - \dfrac{\beta\delta}{1-\beta\delta}\rho - \ell^*\delta, \\ \hat{a}_2^* = \dfrac{1}{2}\left(\hat{b}^* + r(\infty) + \eta - \sqrt{[r(\infty) - \hat{b}^*]^2 + 2\eta[r(\infty) + \hat{b}^*] + \eta^2}\right), \end{cases} \tag{61}$$

where

$$\hat{b}^* = \frac{a_2(1-a_1)}{1-a_1a_2}r(\infty) - a_2\eta - \frac{a_2\ell^*\delta}{1-a_1a_2}$$

and

$$\eta = \eta(\rho, \delta, \beta) = (\frac{\rho\beta\delta}{(1-a_1a_2)(1-\beta\delta)}.$$

Therefore, this and (52) show that there exist a positive integer N and $t_2 > t_1$ such that for

$$\hat{v}_i^{(n)}(t) > \hat{a}_i^* - \delta \quad \text{for } t > t_2, \; n \geq N, \; i = 1, 2. \tag{62}$$

Clearly, if

$$\begin{aligned}&\ell + \widetilde{\sigma}(\mu_1) + \psi(\mu_1)(t-t_0) + NL\sqrt{4d_i(t-t_1)} \\ &\quad \leq x \leq \ell + \widehat{\sigma}(\mu_2) + \psi(\mu_2)(t_1-t_0) + 3\pi/\gamma - NL\sqrt{4d_i(t-t_1)}, \; i = 1, 2,\end{aligned} \tag{63}$$

then (38) holds and (50) with n replaced by N also holds. Choose $t_1 = ml + t_0$ and $t - t_1 = l$, where $m > 1$, and m and l are both sufficiently large. Then we can rewrite (63) as

$$\begin{aligned} &\ell + \widetilde{\sigma}(\mu_1) + \psi(\mu_1)l(m+1) + NL\sqrt{4d_i l} \\ &\qquad \le x \le \ell + \widehat{\sigma}(\mu_2) + ml\psi(\mu_2) + 3\pi/\gamma - NL\sqrt{4d_i l}, \end{aligned} \tag{64}$$

that is,

$$\begin{aligned} &(t_0 + l(m+1))\left[\psi(\mu_1) + \frac{\ell + \widetilde{\sigma}(\mu_1)}{l(m+1)} + \frac{NL\sqrt{4d_i}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0 + l(m+1)} \le x \le \\ &(t_0 + l(m+1))\left[\psi(\mu_2) - \frac{1}{m+1}\psi(\mu_2) + \frac{\ell + \widehat{\sigma}(\mu_2) + 3\pi/\gamma}{l(m+1)} - \frac{NL\sqrt{4d_i}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0 + l(m+1)}, \end{aligned} \tag{65}$$

for $i = 1, 2$. Now for given ε satisfying ε with $0 < \varepsilon < (\hat{c}^*(\infty) - c)/2$, choose δ sufficiently small such that $\delta < \varepsilon/3$. Then there exist l_1 and m_1 sufficiently large such that for $m > m_1$, $l > l_1$ and $t = t_0 + l(m+1) > t_2$, we have

$$\begin{aligned} &(t_0 + l(m+1))\left[\psi(\mu_1) + \frac{\ell + \widetilde{\sigma}(\mu_1)}{l(m+1)} + \frac{NL\sqrt{4d_i}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0 + l(m+1)} \\ <& t(\psi(\mu_1) + \delta) = t(c + \delta + \delta) \\ <& t(c + \varepsilon) \end{aligned}$$

and

$$\begin{aligned} &(t_0 + l(m+1))\left[\psi(\mu_2) - \frac{1}{m+1}\psi(\mu_2) + \frac{\ell + \widehat{\sigma}(\mu_2) + 3\pi/\gamma}{l(m+1)} - \frac{NL\sqrt{4d_i}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0 + l(m+1)} \\ >& t(\psi(\mu_2) - \delta) = t(\hat{c}^*(\infty) - 2\delta - \delta) \\ >& t(\hat{c}^*(\infty) - \varepsilon), \quad i = 1, 2. \end{aligned}$$

Let $t_3 = t_0 + l_1(m_1 + 1)$. If $t > t_3$, then $t(c + \varepsilon) \le x \le t(\hat{c}^*(\infty) - \varepsilon)$ implies that (63) holds. Thus, by (49), (55) and (60), we have that

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)} v_i(t,x)\right] \ge \hat{a}_i^*, \quad i = 1, 2. \tag{66}$$

Because $\delta > 0$ can be arbitrarily small and (61), we have actually shown that

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)} v_1(t,x)\right] \ge \frac{1 - a_1}{1 - a_1 a_2} r(\infty) \tag{67}$$

and

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)} v_2(t,x)\right] \ge \frac{a_2(1 - a_1)}{1 - a_1 a_2} r(\infty). \tag{68}$$

Let $v^*(t,x) = v^*(t,x,\phi^*)$ be the solution of (22) with $\phi_1^*(x) = r(\infty) - \theta_1(x) = r(\infty) - \phi_1(x)$ and $\phi_2^* = r(\infty) - \theta_2(x) = \phi_2(x)$. By similar arguments (symmetry indeed), we obtain the following inequalities which are parallel to (67) and (68):

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)} v_1^*(t,x)\right] \ge \frac{a_1(1 - a_2)}{1 - a_1 a_2} r(\infty) \tag{69}$$

and

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)} v_2^*(t,x)\right] \ge \frac{1 - a_2}{1 - a_1 a_2} r(\infty). \tag{70}$$

By (23), we can rewrite (69) and (70) as

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}(r(\infty)-v_1(t,x))\right]\ge\frac{a_1(1-a_2)}{1-a_1a_2}r(\infty)$$

and

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}(r(\infty)-v_2(t,x))\right]\ge\frac{1-a_2}{1-a_1a_2}r(\infty),$$

respectively, which are equivalent to

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}v_1(t,x)\right]\le\frac{1-a_1}{1-a_1a_2}r(\infty) \tag{71}$$

and

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}v_2(t,x)\right]\le\frac{a_2(1-a_1)}{1-a_1a_2}r(\infty). \tag{72}$$

It follows from (67), (68), (71) and (72) that

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}\left|\frac{1-a_1}{1-a_1a_2}r(\infty)-v_1(t,x)\right|\right]=0$$

and

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}\left|\frac{a_2(1-a_1)}{1-a_1a_2}r(\infty)-v_2(t,x)\right|\right]=0.$$

These two equations together with (10) lead to

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}\left|\frac{1-a_1}{1-a_1a_2}r(\infty)-u_1(t,x,\phi)\right|\right]=0 \tag{73}$$

and

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(\hat{c}^*(\infty)-\varepsilon)}\left|\frac{1-a_2}{1-a_1a_2}r(\infty)-u_2(t,x,\phi)\right|\right]=0. \tag{74}$$

The proof is completed. □

Remark 1. Theorems 2.1, 2.2 and 2.7 imply that in the case of *weak competition*, two competition species will coexist if $0<c<\hat{c}^*(\infty)$; species 1 will become extinct in the habitat and species 2 will persist and spread if $c_1^*(\infty)<c<c_2^*(\infty)$, and the two competing species will both go extinct in the habitat if $c>c_2^*(\infty)$.

Remark 2. Theorem 2.7 implies that if the environment worsening speed c is less than $\hat{c}^*(\infty)$ (a parameter reflecting the two species' individual spreading capability and their competition strengths), then the two species can co-persist by spreading to the right at certain speeds $\hat{c}_1$ and $\hat{c}_2$ respectively, where, by Theorems 2.3 and 2.7, $\hat{c}_i\in[\hat{c}^*(\infty),c_i^*(\infty)]\subset[c,c_i^*(\infty)]$. Here the condition $c<\hat{c}^*(\infty)$ is *only a sufficient condition* for co-persistence; some numerical simulations presented in Section 3 (see Figure 4) shows that it is *not a necessary condition* for co-persistence.

Remark 3. We point out that in the case of *strong competition* (i.e.,the case of $a_1\ge1$ and $a_2\ge1$), the functions $\widetilde{v}(\mu;t,x)$ and $\widehat{v}(\mu;t,x)$ constructed in Lemma 2.6 are *no longer* continuous weak upper and lower solutions of (11), and hence, the above fluctuation method does not seem to apply (at least directly) in the case of strong competition. It remains an interesting and challenging problem to explore the spatial dynamics of model (5) subject to the strong competition. In the next subsection, we explore an alternative approach to deal with this case.

2.2. **The case involving strong competition.** In this subsection, we consider the case of strong competition, i.e., the case of $a_1 \geq 1$ and $a_2 \geq 1$ in (6) with the scenario $0 < c < c_1^*(\infty)$ $(< c_2^*(\infty))$. In this case, $\hat{c}^*(\infty)$ is no longer defined.

Note that the model (6) is heterogeneous in space and time. Due to heterogeneity described by $r(x-ct)$, the existing theory and results on spreading speeds of diffusive *strong competition* system with *constant growth rate* (see e.g., [8] and the references therein) are not (at least directly) applicable to (6). We also point out the fluctuation method used in Subsection 2.1 does not seem to apply in the case of *strong competition*. In this subsection, we develop a new approach that enables us to obtain the spatial dynamics of the model (6) subject to the strong competition. More specifically, we firstly design a subtle iteration scheme which will generate a sequence $\{u^{(k)}\}_{k=0}^{\infty} = \{(u_1^{(k)}, u_2^{(k)}\}_{k=0}^{\infty}$ functions satisfying some required properties. Then, by carefully analyzing this sequence, and with the help of Egorov's Theorem, we show that this sequence converge to a limit function u^*, uniformly on every bounded subset of $\mathbb{R}_+ \times \mathbb{R}$, implying that u^* is the solution to (6) which also satisfies the required properties.

For convenience, we denote by c_1^* and c_2^* the constants $c_1^*(\infty)$ and $c_2^*(\infty)$, respectively. Consider the following system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x-ct) - u_1 - a_1 u_2], \ t>0, \ x \in \mathbb{R}, \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x-ct) - u_2], \ t>0, \ x \in \mathbb{R}, \\ u(0,x) = \phi(x), \ x \in \mathbb{R}, \end{cases} \tag{75}$$

where $\vec{0} \leq \phi(x) \leq \vec{r}(\infty)$, $\phi(x) \equiv \vec{0}$ for all sufficiently large x and $\phi(x) > \vec{0}$ on a closed interval. We denote by $(u_1^{(0)}(t,x,\phi), u_2^{(0)}(t,x,\phi))$ the solution of the system (75). Then by Theorem 2.2 in [27], for any $\varepsilon \in (0, (c_2^* - c)/2)$, $u_2^{(0)}(t,x,\phi)$ satisfies

$$\begin{aligned} &\lim_{t\to\infty}\left[\sup_{x \leq t(c-\varepsilon)} u_2^{(0)}(t,x,\phi)\right] = 0, \\ &\lim_{t\to\infty}\left[\sup_{x \geq t(c_2^*+\varepsilon)} u_2^{(0)}(t,x,\phi)\right] = 0, \\ &\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\leq x \leq t(c_2^*-\varepsilon)} \left| r(\infty) - u_2^{(0)}(t,x,\phi)\right|\right] = 0. \end{aligned} \tag{76}$$

Furthermore, $u_1^{(0)}(t,x,\phi)$ enjoys the following property.

Lemma 2.8. *For every $\delta > 0$, there exists $T > 0$ such that $u_1^{(0)}(t,x,\phi) \leq \delta$ for all $x \in \mathbb{R}$ and $t \geq T$.*

Proof. Without loss of generality, we can assume $\delta \in (0,1)$. Let $\rho = \delta/30$. Then clearly $u_1^{(0)}(t,x,\phi)$ satisfies the integral equation

$$\begin{aligned} u_1^{(0)}(t,x,\phi) = &\int_{-\infty}^{+\infty} e^{-\rho t} k_{11}(t, x-y)\phi_1(y)dy \\ &+ \int_0^t e^{-\rho s} \int_{-\infty}^{+\infty} k_{11}(s,y) h_1^{(0)}(\rho, t-s, x-y) dy ds, \end{aligned} \tag{77}$$

where

$$k_{11}(s,y)=\frac{1}{\sqrt{4\pi d_1 s}}e^{-\frac{y^2}{4d_1 s}} \tag{78}$$

and

$$h_1^{(0)}(\rho,s,y)=u_1^{(0)}(s,y,\phi)\left[\rho+r(y-cs)-u_1^{(0)}(s,y,\phi)-a_1u_2^{(0)}(s,y,\phi)\right]. \tag{79}$$

Note that $\int_{-\infty}^{+\infty}k_{11}(s,y)dy=1$. By a comparison argument and condition (A), we know that $0\le u_i^{(0)}(t,x,\phi)\le r(\infty)$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ and $i=1,2$, and hence,

$$h_1^{(0)}(\rho,t,x)\le r(\infty)(\rho+2r(\infty)+a_1r(\infty)),\quad \forall(t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{80}$$

On the other hand, since $\int_0^{+\infty}e^{-\rho s}ds$ is convergent, for the above $\delta>0$, there exist $\eta>0$ and $A>\eta$ such that

$$\int_0^{\eta}e^{-\rho s}\int_{-\infty}^{+\infty}k_{11}(s,y)h_1^{(0)}(\rho,t-s,x-y)dyds<\frac{\delta}{10},\quad \forall(t,x)\in\mathbb{R}_+\times\mathbb{R}, \tag{81}$$

and

$$\int_A^{+\infty}e^{-\rho s}\int_{-\infty}^{+\infty}k_{11}(s,y)h_1^{(0)}(\rho,t-s,x-y)dyds<\frac{\delta}{10},\quad \forall(t,x)\in\mathbb{R}_+\times\mathbb{R}. \tag{82}$$

By (75) and [27], For above $\delta>0$ and for any ε with $0<\varepsilon<(c_2^*-c)/4$, there exists $T_0>0$ such that

$$\int_{-\infty}^{+\infty}e^{-\rho t}k_{11}(t,x-y)\phi_1(y)dy<\delta/5,\quad \forall(t,x)\in[T_0,+\infty)\times\mathbb{R} \tag{83}$$

and furthermore by (76) and $a_1\ge 1$,

$$\begin{aligned}&r(y-cs)-a_1u_2^{(0)}(s,y,\phi)<r(\infty)-u_2^{(0)}(s,y,\phi)\\&<\frac{\delta}{30},\qquad \forall(s,y)\in\{(s,y)|s\ge T_0,s(c+\varepsilon)\le y\le s(c_2^*-\varepsilon)\}.\end{aligned}$$

We claim that

$$h_1^{(0)}(\rho,t,x)<\frac{\delta^2}{150},\quad \forall(t,x)\in\{(s,y)|s\ge T_0,s(c+\varepsilon)\le y\le s(c_2^*-\varepsilon)\}. \tag{84}$$

In fact, if $u_1^{(0)}(t,x,\phi)<\delta/10$ and $(t,x)\in\{(s,y)|s\ge T_0,s(c+\varepsilon)\le y\le s(c_2^*-\varepsilon)\}$, then

$$h_1^{(0)}(\rho,t,x)<\frac{\delta}{10}\left(\rho+\frac{\delta}{30}\right)=\frac{\delta^2}{150};$$

and if $u_1^{(0)}(t,x,\phi)\ge\frac{\delta}{10}$ and $(t,x)\in\{(s,y)|s\ge T_0,s(c+\varepsilon)\le y\le s(c_2^*-\varepsilon)\}$, then

$$h_1^{(0)}(\rho,t,x)\le u_1^{(0)}(t,x,\phi)\left(\frac{\delta}{30}+\frac{\delta}{30}-\frac{\delta}{10}\right)\le 0<\frac{\delta^2}{150}.$$

Thus, (84) holds true.

For the above $\varepsilon>0$, we write

$$\int_{\eta}^{A}e^{-\rho s}\int_{-\infty}^{+\infty}k_{11}(s,y)h_1^{(0)}(\rho,t-s,x-y)dyds=\sum_{i=1}^{3}I_i(\varepsilon,t,x), \tag{85}$$

where

$$I_1(\varepsilon,t,x)=\int_{\eta}^{A}e^{-\rho s}\int_{-\infty}^{x-(c_2^*-\varepsilon)(t-s)}k_{11}(s,y)h_1^{(0)}(\rho,t-s,x-y)dyds,$$

$$I_2(\varepsilon,t,x)=\int_\eta^A e^{-\rho s}\int_{x-(c_2^*-\varepsilon)(t-s)}^{x-(c+\varepsilon)(t-s)} k_{11}(s,y)h_1^{(0)}(\rho,t-s,x-y)dyds,$$

and

$$I_3(\varepsilon,t,x)=\int_\eta^A e^{-\rho s}\int_{x-(c+\varepsilon)(t-s)}^{+\infty} k_{11}(s,y)h_1^{(0)}(\rho,t-s,x-y)dyds,$$

For $I_1(\varepsilon,t,x)$, when $y\le x-(c_2^*-\varepsilon)(t-s)$ with $x\le(c_2^*-2\varepsilon)t$, we have

$$y\le(c_2^*-2\varepsilon)t-(c_2^*-\varepsilon)(t-s)=-\varepsilon t+(c_2^*-\varepsilon)s.$$

Therefore, by (78) and (80), for all $x\le(c_2^*-2\varepsilon)t$ and $t>Ac_2^*/\varepsilon$, we have

$$\begin{aligned}
I_1(\varepsilon,t,x)&\le r(\infty)[\rho+2r(\infty)+a_1r(\infty)]\int_\eta^A e^{-\rho s}\int_{-\infty}^{x-(c_2^*-\varepsilon)(t-s)} k_{11}(s,y)dyds\\
&\le r(\infty)[\rho+2r(\infty)+a_1r(\infty)]\int_\eta^A e^{-\rho s}\int_{-\infty}^{-\varepsilon t+(c_2^*-\varepsilon)s} k_{11}(s,y)dyds\\
&\le r(\infty)[\rho+2r(\infty)+a_1r(\infty)]\int_\eta^A e^{-\rho s}\int_{-\infty}^{-\varepsilon t+(c_2^*-\varepsilon)A} k_{11}(s,y)dyds\\
&\le r(\infty)[\rho+2r(\infty)+a_1r(\infty)]\int_\eta^A e^{-\rho s}\int_{-\infty}^{[(c_2^*-\varepsilon)A-\varepsilon t]/\sqrt{4d_1A}} \frac{1}{\sqrt{\pi}}e^{-z^2}dzds\\
&\le r(\infty)[\rho+2r(\infty)+a_1r(\infty)]\frac{1}{\rho\sqrt{\pi}}\int_{-\infty}^{[(c_2^*-\varepsilon)A-\varepsilon t]/\sqrt{4d_1A}} e^{-z^2}dz.
\end{aligned}$$

Obviously,

$$\lim_{t\to+\infty}\frac{(c_2^*-\varepsilon)A-\varepsilon t}{\sqrt{4d_1A}}=-\infty.$$

Thus, for the above $\delta>0$, there exists $t_1>\max\{T_0,Ac_2^*/\varepsilon\}$ such that

$$I_1(\varepsilon,t,x)<\delta/5\quad\text{for all } x\le(c_2^*-2\varepsilon)t \text{ and } t>t_1. \tag{86}$$

For $I_2(\varepsilon,t,x)$, when $x-(c_2^*-\varepsilon)(t-s)\le y\le x-(c+\varepsilon)(t-s)$, there holds

$$(c+\varepsilon)(t-s)\le x-y\le(c_2^*-\varepsilon)(t-s).$$

Therefore, it follows from (84) that

$$h_1^{(0)}(\rho,t-s,x-y)<\frac{\delta^2}{150}\ \text{ for all }\ x-(c_2^*-\varepsilon)(t-s)\le y\le x-(c+\varepsilon)(t-s)\ \ t\ge T_0,$$

where $s\in[\eta,A]$. Thus, we obtain that

$$\begin{aligned}
I_2(\varepsilon,t,x)&<\frac{\delta^2}{150}\int_\eta^A e^{-\rho s}\int_{x-(c_2^*-\varepsilon)(t-s)}^{x-(c+\varepsilon)(t-s)} k_{11}(s,y)dyds\\
&<\frac{\delta^2}{150\rho}=\frac{\delta}{5}\ \text{ for all }\ (t,x)\in[T_0,+\infty)\times\mathbb{R}.
\end{aligned}\tag{87}$$

For $I_3(\varepsilon,t,x)$, when $y\ge x-(c+\varepsilon)(t-s)$ with $x\ge t(c+2\varepsilon)$, there holds

$$y\ge t(c+2\varepsilon)-(c+\varepsilon)(t-s)=\varepsilon t+(c+\varepsilon)s.$$

Therefore, by (78) and (80), for all $x \geq (c+2\varepsilon)t$, we have

$$\begin{aligned} I_3(\varepsilon,t,x) &\leq r(\infty)(\rho+2r(\infty)+a_1r(\infty))\int_\eta^A e^{-\rho s}\int_{x-(c+\varepsilon)(t-s)}^{+\infty} k_{11}(s,y)dyds \\ &\leq r(\infty)(\rho+2r(\infty)+a_1r(\infty))\int_\eta^A e^{-\rho s}\int_{\varepsilon t+(c+\varepsilon)s}^{+\infty} k_{11}(s,y)dyds \\ &\leq r(\infty)(\rho+2r(\infty)+a_1r(\infty))\int_\eta^A e^{-\rho s}\int_{\varepsilon t+(c+\varepsilon)\eta}^{+\infty} k_{11}(s,y)dyds \\ &\leq r(\infty)(\rho+2r(\infty)+a_1r(\infty))\int_\eta^A e^{-\rho s}\int_{[\varepsilon t+(c+\varepsilon)\eta]/\sqrt{4d_1A}}^{+\infty} \frac{1}{\sqrt{\pi}}e^{-z^2}dzds \\ &\leq r(\infty)(\rho+2r(\infty)+a_1r(\infty))\frac{1}{\rho\sqrt{\pi}}\int_{[\varepsilon t+(c+\varepsilon)\eta]/\sqrt{4d_1A}}^{+\infty} e^{-z^2}dz. \end{aligned} \tag{88}$$

Since

$$\lim_{t\to+\infty}\frac{\varepsilon t+(c+\varepsilon)\eta}{\sqrt{4d_1A}} = +\infty,$$

there exists $t_2 > t_1$ such that

$$I_3(\varepsilon,t,x) < \delta/5 \ \text{ for all } \ x \geq (c+2\varepsilon)t \ \text{ and } \ t > t_2. \tag{89}$$

Thus, it follows from (77), (81), (82), (83), (85), (86), (87) and (89) that

$$u_1^{(0)}(t,x,\phi) < \delta \ \text{ for } \ (t,x) \in \{(t,x)\,|t \geq t_2, (c+2\varepsilon)t \leq x \leq (c_2^*-2\varepsilon)t\},$$

which implies

$$\lim_{t\to+\infty}\left[\sup_{(c+2\varepsilon)t\leq x\leq t(c_2^*-2\varepsilon)} u_1^{(0)}(t,x,\phi)\right] = 0.$$

By [27] and the comparison principle, for the above $\varepsilon > 0$,

$$\lim_{t\to+\infty}\left[\sup_{x\leq(c-\varepsilon)t} u_1^{(0)}(t,x,\phi)\right] = 0$$

and

$$\lim_{t\to+\infty}\left[\sup_{x\geq(c_1^*+\varepsilon)t} u_1^{(0)}(t,x,\phi)\right] = 0.$$

Because $\varepsilon > 0$ is arbitrary, we have actually shown that

$$\lim_{t\to+\infty}\left[\sup_{x\in\mathbb{R}} u_1^{(0)}(t,x,\phi)\right] = 0.$$

The proof is completed. □

Remark 4. From the proof of Lemma 2.8, we easily see that in the case of strong competition, the conclusion of Lemma 2.8 remains valid even if $c \geq c_1^*(\infty)$ (assuming $c < c_2^*$). Also, we have seen that we have only used the condition $a_1 \geq 1$, meaning that $a_2 \geq 1$ is actually *not* required.

Similarly, we consider the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x-ct) - u_1 - a_1 u_2],\ t>0,\ x \in \mathbb{R}, \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x-ct) - u_2 - a_2 u_1^{(0)}],\ t>0,\ x \in \mathbb{R}, \\ u(0,x) = \phi(x),\ x \in \mathbb{R}. \end{cases} \tag{90}$$

Let $(u_1^{(1)}(t,x,\phi), u_2^{(1)}(t,x,\phi))$ be the solution of the system (90). Then by $u_1^{(0)} \geq 0$ and the comparison principle, we obtain that $u_2^{(1)} \leq u_2^{(0)}$ and $u_1^{(1)} \geq u_1^{(0)}$. And by an comparison argument and (76) (also see [27]), for any $\varepsilon > 0$ there are hold

$$\lim_{t\to\infty} \left[\sup_{x \leq t(c-\varepsilon)} u_2^{(1)}(t,x,\phi) \right] = 0 \tag{91}$$

and

$$\lim_{t\to\infty} \left[\sup_{x \geq t(c_2^*+\varepsilon)} u_2^{(1)}(t,x,\phi) \right] = 0. \tag{92}$$

Moreover, by making use of Lemma 2.8, we can establish the following result.

Lemma 2.9. *For every* ε *with* $0 < \varepsilon < (c_2^* - c)/2$,

$$\lim_{t\to\infty} \left[\sup_{t(c+\varepsilon) \leq x \leq t(c_2^*-\varepsilon)} \left| r(\infty) - u_2^{(1)}(t,x,\phi) \right| \right] = 0.$$

Proof. By Lemma 2.8, for any δ with $0 < \delta < r(\infty)/a_2$, there exists $T > 0$ such that

$$u_1^{(0)}(t,x,\phi) < \delta \tag{93}$$

for all $(t,x) \in [T,+\infty) \times \mathbb{R}$. Let $r_\delta(x-ct) = r(x-ct) - a_2\delta$. Then $r_\delta(x)$ is continuous, nondecreasing and bounded, and piecewise continuously differentiable in x for $x \in \mathbb{R}$ with $-\infty < r_\delta(-\infty) < 0 < r_\delta(\infty) < \infty$. Let $\hat{u}_2^{(1)}(t,x,\phi)$ be the solution of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_2 \dfrac{\partial^2 u}{\partial x^2} + u[r_\delta(x-ct) - u],\ t > T,\ x \in \mathbb{R}, \\ u(T,x) = u_2^{(1)}(T,x,\phi),\ x \in \mathbb{R}. \end{cases} \tag{94}$$

Since $\phi_2(x) > 0$ on a closed interval, $u_2^{(1)}(T,x,\phi)$ is also positive on some closed interval. Thus, it follows from (90), (93), (94), the comparison principle and Theorem 2.2-(iii) in [27] that for every ε with $0 < \varepsilon < (c_\delta^* - c)/2$,

$$\lim_{t\to\infty} \left[\inf_{t(c+\varepsilon) \leq x \leq t(c_\delta^*-\varepsilon)} u_2^{(1)}(t,x,\phi) \right] \geq \lim_{t\to\infty} \left[\inf_{t(c+\varepsilon) \leq x \leq t(c_\delta^*-\varepsilon)} \hat{u}_2^{(1)}(t,x,\phi) \right] \geq r_\delta(\infty),$$

where $c_\delta^* = 2\sqrt{d_2 r_\delta(\infty)}$. Because δ is arbitrary, we have actually shown that for any ε with $0 < \varepsilon < (c_2^* - c)/2$,

$$\lim_{t\to\infty} \left[\inf_{t(c+\varepsilon) \leq x \leq t(c_2^*-\varepsilon)} u_2^{(1)}(t,x,\phi) \right] \geq r(\infty).$$

This together with the fact that $u_2^{(1)}(t,x,\phi) \leq r(\infty)$ implies

$$\lim_{t\to\infty} \left[\sup_{t(c+\varepsilon) \leq x \leq t(c_2^*-\varepsilon)} \left| r(\infty) - u_2^{(1)}(t,x,\phi) \right| \right] = 0,$$

completing the proof. □

Remark 5. Remark 4 and the proof of Lemma 2.9 imply that in the case of strong competition, even if $c_1^*(\infty) \le c < c_2^*(\infty)$ and $a_2 < 1$, the conclusion of Lemma 2.9 still holds.

Obviously, by employing (90), (91), (92), Lemma 2.9 and the same arguments as in the proof of Lemma 2.8, we have the following lemma.

Lemma 2.10. *For every $\varepsilon > 0$, there exists $T > 0$ such that $u_1^{(1)}(t,x,\phi) \le \varepsilon$ for all $x \in \mathbb{R}$ and $t \ge T$.*

Lemmas 2.8-2.10 motivates us to consider the following iteration scheme:

$$\begin{cases} \dfrac{\partial u_1^{(k)}}{\partial t} = d_1 \dfrac{\partial^2 u_1^{(k)}}{\partial x^2} + u_1^{(k)}[r(x-ct) - u_1^{(k)} - a_1 u_2^{(k)}],\ t > 0,\ x \in \mathbb{R}, \\ \dfrac{\partial u_2^{(k)}}{\partial t} = d_2 \dfrac{\partial^2 u_2^{(k)}}{\partial x^2} + u_2^{(k)}[r(x-ct) - u_2^{(k)} - a_2 u_1^{(k-1)}],\ t > 0,\ x \in \mathbb{R}, \\ u^{(k)}(0,x) = \phi(x),\ x \in \mathbb{R}, \\ k = 1, 2, \cdots. \end{cases} \tag{95}$$

With $(u_1^{(0)}, u_2^{(0)})$ being the solution of the system (75), this iteration generates a sequence $\{(u_1^{(k)}, u_2^{(k)})\}_{k=0}^{\infty}$ of functions. By Lemmas 2.8-2.10, this sequence obviously satisfies the following properties:

(i) the sequence $\left\{u_1^{(k)}\right\}_{k=0}^{\infty}$ is nondecreasing and the sequence $\left\{u_2^{(k)}\right\}_{k=0}^{\infty}$ nonincreasing:

$$0 \le u_1^{(0)} \le u_1^{(1)} \le \cdots \le u_1^{(k)} \le u_1^{(k+1)} \le \cdots \le r(\infty) \tag{96}$$

and

$$r(\infty) \ge u_2^{(0)} \ge u_2^{(1)} \ge \cdots \ge u_2^{(k)} \ge u_2^{(k+1)} \ge \cdots \ge 0; \tag{97}$$

(ii) for any $\varepsilon > 0$ and $k = 0, 1, 2, \cdots$,

$$\lim_{t\to\infty} \left[\sup_{x \le t(c-\varepsilon)} u_2^{(k)}(t,x,\phi) \right] = 0, \tag{98}$$

$$\lim_{t\to\infty} \left[\sup_{x \ge t(c_2^*(\infty)+\varepsilon)} u_2^{(k)}(t,x,\phi) \right] = 0, \tag{99}$$

$$\lim_{t\to\infty} \left[\sup_{t(c+\varepsilon) \le x \le t(c_2^*(\infty)-\varepsilon)} \left| r(\infty) - u_2^{(k)}(t,x,\phi) \right| \right] = 0 \text{ when } 0 < \varepsilon < (c_2^* - c)/2, \tag{100}$$

and

$$\lim_{t\to\infty} \left[\sup_{x\in\mathbb{R}} u_1^{(k)}(t,x,\phi) \right] = 0. \tag{101}$$

By (i), $\{u_1^{(k)}\}_{k=0}^{\infty}$ and $\{u_2^{(k)}\}_{k=0}^{\infty}$ both converge pointwise, as $k \to \infty$, that is, there are $u_1^*(t,x,\phi)$ and $u_2^*(t,x,\phi)$ such that

$$\lim_{k\to\infty} u_1^{(k)}(t,x,\phi) = u_1^*(t,x,\phi) \text{ and } \lim_{k\to\infty} u_2^{(k)}(t,x,\phi) = u_2^*(t,x,\phi) \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{102}$$

Now we are in a position to prove our main result in the case of strong competition.

Theorem 2.11. *Assume that (A) holds, $a_j \geq 1$ for $j = 1, 2$, and $0 < c < c_1^*(\infty)$. Let $u^*(t,x,\phi) = (u_1^*(t,x,\phi), u_2^*(t,x,\phi))$, where $u_1^*(t,x,\phi)$ and $u_2^*(t,x,\phi)$ are defined in (102). Then $u^*(t,x,\phi)$ is the solution of (6) with $\vec{0} \leq \phi(x) \leq \vec{r}(\infty)$ and the following statements hold.*

- **(i)** *If $\phi_1(x) \equiv 0$ for all sufficiently large x, then for every $\delta > 0$ there exits $t_0 > 0$ such that $u_1^*(t,x,\phi) \leq \varepsilon$ for all $(t,x) \in [t_0, +\infty) \times \mathbb{R}$;*
- **(ii)** *For any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x \leq t(c-\varepsilon)} u_2^*(t,x,\phi)\right] = 0;$$

- **(iii)** *If $\phi_2(x) \equiv 0$ for all sufficiently large x, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x \geq t(c_2^*+\varepsilon)} u_2^*(t,x,\phi)\right] = 0;$$

- **(iv)** *If $\phi_1(x) \equiv 0$ for all sufficiently large x and $\phi_2(x) > 0$ on a closed interval, then for any ε with $0 < \varepsilon < (c_2^* - c)/2$,*

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\leq x \leq t(c_2^*-\varepsilon)} |r(\infty) - u_2^*(t,x,\phi)|\right] = 0.$$

Proof. Denote by z the vector (t,x). For any given $T > 0$ and $M > 0$, let $\Lambda = [0,T] \times [-M, M]$. We first show that the convergence in (102) is uniform for $z \in \Lambda$. Indeed, by (95), we can obtain that

$$u_1^{(k)}(z,\phi) - u_1^{(k+p)}(z,\phi) = \int_0^t \int_{-\infty}^{+\infty} k_{11}(t-s, x-y) h_1(s,y,k,p)\,dyds, \quad \forall z \in \Lambda, \tag{103}$$

where

$$\begin{aligned} h_1(s,y,k,p) = &\left[u_1^{(k+p)}(s,y) + u_1^{(k)}(s,y) + a_1 u_2^{(k)}(s,y) - r(y-cs)\right] \\ &\times \left[u_1^{(k+p)}(s,y) - u_1^{(k)}(s,y)\right] + a_1 u_1^{(k+p)}(s,y)\left[u_2^{(k+p)}(s,y) - u_2^{(k)}(s,y)\right]. \end{aligned} \tag{104}$$

Let $\widetilde{h}_1 = 2(3 + 2a_1)r^2(\infty)$. Then $|h_1(s,y,k,p)| \leq \widetilde{h}_1$ for all $s \in \mathbb{R}_+$, $y \in \mathbb{R}$, $k = 0, 1, 2, \cdots$ and $p = 1, 2, \cdots$.

For any given $\varepsilon > 0$, there exists $L > 0$ such that

$$\int_{-L}^{L} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \geq 1 - \frac{\varepsilon}{5T\widetilde{h}_1}.$$

Therefore, for any $s > 0$,

$$\int_{-L\sqrt{4d_1 s}}^{L\sqrt{4d_1 s}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{x^2}{4d_1 s}} dx \geq 1 - \frac{\varepsilon}{5T\widetilde{h}_1}.$$

It is clear from (103) that

$$\begin{aligned} &\left|u_1^{(k)}(z,\phi) - u_1^{(k+p)}(z,\phi)\right| \\ = &\left|\int_0^t \int_{-\infty}^{+\infty} k_{11}(s,y) h_1(t-s, x-y, k, p)\,dyds\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{5} + \left| \int_0^t \int_{-L\sqrt{4d_1 s}}^{L\sqrt{4d_1 s}} k_{11}(s,y) h_1(t-s, x-y, k, p) dyds \right| \\
&= \frac{\varepsilon}{5} + \left| \int_0^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, k, p) dyds \right|, \ \forall z \in \Lambda.
\end{aligned}
$$

Let $\delta_1 = \varepsilon / 5\widetilde{h}_1$. In the case of $t \leq \delta_1$, we have

$$
\begin{aligned}
&\left| u_1^{(k)}(z,\phi) - u_1^{(k+p)}(z,\phi) \right| \\
&\leq \frac{\varepsilon}{5} + \int_0^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) \left| h_1(s,y,k,p) \right| dyds \\
&\leq \frac{\varepsilon}{5} + \widetilde{h}_1 \int_0^{\delta_1} \int_{-\infty}^{+\infty} k_{11}(s,y) dyds \\
&= \frac{2\varepsilon}{5}, \ \forall z \in \Lambda.
\end{aligned}
$$

In the case of $t > \delta_1$, we obtain that

$$
\begin{aligned}
&\left| \int_{t-\delta_1}^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, k, p) dyds \right| \\
\leq &\int_0^{\delta_1} \int_{-\infty}^{+\infty} k_{11}(s,y) \left| h_1(t-s, x-y, k, p) \right| dyds \\
\leq &\widetilde{h}_1 \int_0^{\delta_1} \int_{-\infty}^{+\infty} k_{11}(s,y) dyds \\
= &\frac{\varepsilon}{5}, \ \forall z \in \Lambda,
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&\left| u_1^{(k)}(z,\phi) - u_1^{(k+p)}(z,\phi) \right| \\
\leq &\frac{\varepsilon}{5} + \left| \int_0^{t-\delta_1} \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, k, p) dyds \right| \\
&+ \left| \int_{t-\delta_1}^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, k, p) dyds \right| \\
\leq &\frac{2\varepsilon}{5} + \left| \int_0^{t-\delta_1} \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, k, p) dyds \right|, \ \forall z \in \Lambda.
\end{aligned}
\tag{105}
$$

In order to estimate the integral on the last line of (105), for given $z \in \Lambda$ we define

$$
\Lambda_z = \left\{ (s,y) | 0 \leq s \leq t - \delta_1, x - L\sqrt{4d_1(t-s)} \leq y \leq x + L\sqrt{4d_1(t-s)} \right\}.
$$

Obviously, $\Lambda_z \subset \Lambda_1$ where

$$
\Lambda_1 = [0,T] \times [-M - L\sqrt{4d_1 T}, M + L\sqrt{4d_1 T}]
$$

is bounded. Thus, by Egorov's Theorem (see, e.g., Theorem 3.2.8 in [48]), for the above $\varepsilon > 0$ there exists a measurable subset Λ_ε of Λ_1 such that $m(\Lambda_1 - \Lambda_\varepsilon) <$

$\sqrt{4\pi d_1}(5\widetilde{h}_1)^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}}$,

$$\lim_{k\to\infty} u_1^{(k)}(z,\phi) = u_1^*(z,\phi) \quad \text{and} \quad \lim_{k\to\infty} u_2^{(k)}(z,\phi) = u_2^*(z,\phi) \tag{106}$$

uniformly for all $z \in \Lambda_\varepsilon$, where $m(\Lambda_1 - \Lambda_\varepsilon)$ is the measure of the set $\Lambda_1 - \Lambda_\varepsilon$. Thus, for the above $\varepsilon > 0$ there exist $K_\varepsilon > 0$ and $P_\varepsilon > 0$ such that

$$\begin{cases} |u_1^{(k+p)}(z,\phi) - u_1^{(k)}(z,\phi)| < \dfrac{2\varepsilon}{5T(3+2a_1)r(\infty)}, \\ |u_2^{(k+p)}(z,\phi) - u_2^{(k)}(z,\phi)| < \dfrac{2\varepsilon}{5T(3+2a_1)r(\infty)}, \end{cases} \text{for } k > K_\varepsilon,\ p > P_\varepsilon,\ z \in \Lambda_\varepsilon. \tag{107}$$

It follows from (104) and (107) that

$$|h_1(s,y,k,p)| \le \frac{2\varepsilon}{5T} \quad \text{for } k > K_\varepsilon,\ p > P_\varepsilon,\ (s,y) \in \Lambda_\varepsilon. \tag{108}$$

By (105) and (108), we obtain that

$$\begin{aligned} &\left|u_1^{(k)}(z,\phi) - u_1^{(k+p)}(z,\phi)\right| \\ \le& \frac{2\varepsilon}{5} + \int\int_{\Lambda_\varepsilon \bigcap \Lambda_z} k_{11}(t-s,x-y)\,|h_1(s,y,k,p)|\,dyds \\ &+ \int\int_{(\Lambda_1-\Lambda_\varepsilon)\bigcap \Lambda_z} k_{11}(t-s,x-y)\,|h_1(s,y,k,p)|\,dyds \\ \le& \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5T}\int\int_{\Lambda_\varepsilon \bigcap \Lambda_z} k_{11}(t-s,x-y)dyds + \widetilde{h}_1 \int\int_{(\Lambda_1-\Lambda_\varepsilon)\bigcap \Lambda_z} k_{11}(t-s,x-y)dyds \\ \le& \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5T}\int_0^t \int_{-\infty}^{+\infty} k_{11}(s,y)dyds + \widetilde{h}_1 \int\int_{\Lambda_2^*} \frac{1}{\sqrt{\pi}} e^{-y^2} dyds \\ \le& \frac{4\varepsilon}{5} + \frac{\widetilde{h}_1}{\sqrt{\pi}} m(\Lambda_2^*) \\ \le& \frac{4\varepsilon}{5} + \frac{\widetilde{h}_1}{\sqrt{4d_1\pi\delta_1}} m(\Lambda_1 - \Lambda_\varepsilon) \\ \le& \varepsilon \quad \text{for } k > K_\varepsilon,\ p > P_\varepsilon,\ z \in \Lambda. \end{aligned} \tag{109}$$

where $\Lambda_2^* = f((\Lambda_1 - \Lambda_\varepsilon)\bigcap \Lambda_z)$, $m(\Lambda_2^*)$ is the measure of the set Λ_2^* and $f : (\Lambda_1 - \Lambda_\varepsilon)\bigcap \Lambda_z \to \mathbb{R}^2$ is a bijective function defined by

$$f(s,y) = (s, \frac{y-x}{\sqrt{4d_1(t-s)}}), \qquad \forall (s,y) \in (\Lambda_1 - \Lambda_\varepsilon)\bigcap \Lambda_z.$$

Thus, the limit $\lim_{k\to\infty} u_1^{(k)}(z,\phi) = u_1^*(z,\phi)$ uniformly holds for all $z \in \Lambda$. Similarly, we obtain that the limit $\lim_{k\to\infty} u_2^{(k)}(z,\phi) = u_2^*(z,\phi)$ uniformly holds for all $z \in \Lambda$. Because T and M are arbitrary, we have actually shown that the function sequence $\{\left(u_1^{(k)}, u_2^{(k)}\right)\}_{k=0}^\infty$ converges to (u_1^*, u_2^*) uniformly on every bounded subset of $\mathbb{R}_+ \times \mathbb{R}$. This together with the iteration scheme (95) (e.g., writing (95) as some integral form and applying the Lebesque's dominated convergence theorem) implies that (u_1^*, u_2^*) is the solution to (6).

By (101), for any $\delta > 0$, there exists a positive constant T_δ such that

$$|u_1^{(k)}(z,\phi)| < \frac{\delta}{2} \text{ for } z \in \Lambda_0 = [T_\delta, T_\delta + \tau] \times [-L, L],\ L > 0,\ \tau > 0. \tag{110}$$

For the above $\delta > 0$, there exists $K_{\delta,\Lambda_0} > 0$ such that

$$|u_1^{(k)}(z,\phi) - u_1^*(z,\phi)| < \frac{\delta}{2} \text{ for } z \in \Lambda_0,\ k > K_{\delta,\Lambda_0}. \tag{111}$$

Thus, for the above $\delta > 0$ there holds

$$|u_1^*(z,\phi)| \le |u_1^{(k)}(z,\phi)| + |u_1^{(k)}(z,\phi) - u_1^*(z,\phi)| < \delta \text{ for all } z \in \Lambda_0.$$

Because τ and L are arbitrary, we have actually shown that

$$|u_1^*(z,\phi)| < \delta \text{ for all } z \in [T_\delta, +\infty) \times \mathbb{R}.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{t\to\infty} \left[\sup_{x\in\mathbb{R}} u_1^*(t,x,\phi)\right] = 0. \tag{112}$$

Applying the similar arguments to (98), (99) and (100), we can confirm (ii), (iii) and (iv) of the theorem. The proof is completed. □

Remark 6. From Remarks 4 - 5 and the proof of Theorem 2.11, we know that the conclusions of Theorem 2.11 remains true even if $c_1^*(\infty) \le c < c_2^*(\infty)$. Moreover, for the two competition coefficients, we have only used the condition $a_1 \ge 1$ in obtaining our results in Theorem 2.11, meaning that the condition $a_2 \ge 1$ is actually *not required*. In other words, the key conditions in our results in this section are (A) conditions on speeds: $c_2^* > c$ and $c_2^* > c_1^*$; and (B) the condition on the competition strength: $a_1 \ge 1$. An explanation for this observation is that the condition $c_2^* > c$ means that the larger spreading speed of species 2 enables it to survive the worsening environment, while the condition $a_1 \ge 1$ shows that species 2 have strong competition against species 1, and it is this strong competition from surviving species 2 that drives the the species 1 to extinction, regardless of whether $c_1^* < c$ or $c_1^* > c$, whether $a_2 < 1$ or $a_2 \ge 1$. It is interesting to ask what happens if $c_2^* > c$ (also $(c_2^* > c_1^*)$ but $a_1 < 1 \le a_2$, a scenario that species 2 can persist by spreading to the right in the absence of species 1 and its spread speed is larger than that of species 1, yet species 2 has competition disadvantage. In such a scenario, which species can persist by spreading and by what speed? Under this scenario, there are two cases: $c_1^* > c$ and $c_1^* < c$. The latter has been covered by Theorems 2.1 and 2.2; for the former, we are not able to answer these questions theoretically at the present, but numerical simulations in Section 3 (see Figure 6) shows that co-persistence is possible by spreading to the right.

3. **Some numeric simulations.** In this section, we present some numerical simulations for model (6) for two purposes: (I) numerically confirm the theoretical results obtained in Section 2; (II) numerically explore some parameter ranges that have not been covered in the results of Section 2, by which we hope to gain some intuition and suggestions for further theoretical investigation of (6). To this end, we choose the growth function

$$r(x) = \frac{1.6}{1 + e^{-0.3x}} - 0.6 \tag{113}$$

and the initial data

$$\phi_1(x) = \begin{cases} 0.8\sin(x-15), & \text{if } 15 \le x \le 15+\pi, \\ 0, & \text{elsewhere} \end{cases} \tag{114}$$

and

$$\phi_2(x) = \begin{cases} 0.4\sin(x-10), & \text{if } 10 \le x \le 10+\pi, \\ 0, & \text{elsewhere.} \end{cases} \tag{115}$$

Firstly, we choose

$$d_1 = 1, \ d_2 = 1.21, \ a_1 = 0.19, \ a_2 = 0.36. \tag{116}$$

Then we can calculate to obtain $c_1^*(\infty) = 2$, $c_2^*(\infty) = 2.2$, $\hat{c}_1^*(\infty) = 1.80$ and $\hat{c}_2^*(\infty) = 1.76$, which give $c^*(\infty) = 2$ and $\hat{c}^*(\infty) = 1.76$, where $c^*(\infty) = \min\{c_1^*(\infty), c_2^*(\infty)\}$.

Now, if $c = 2.21$, then, $c > c_i^*(\infty) > \hat{c}_i^*(\infty)$ for $i = 1, 2$, a scenario that the environment is worsening very fast. Not surprisingly, the two species will eventually go to extinction, as claimed in Theorem 2.1 and illustrated in Figure 1.

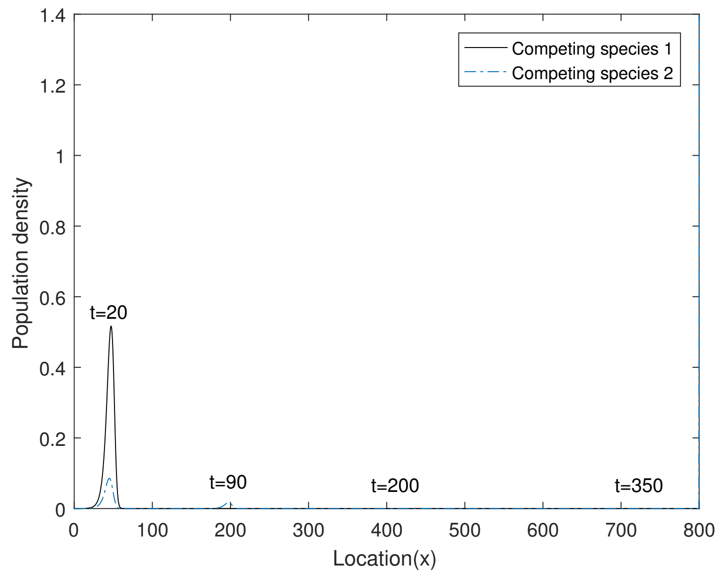

FIGURE 1. Numerical simulations on (6) with (113)-(116): when the environment worsening rate is too large ($c = 2.21 > c_i^*(\infty) > \hat{c}_i^*(\infty)$ for $i = 1, 2$), both species go to extinct in the habitat.

Next, we consider a case that worsening speed c is a little bit slower: $c = 2.05 \in (c_1^*(\infty), c_2^*(\infty))$, a scenario that the spreading capability of species 1 without competition is not enough to allow this species to survive the environment worsening speed, but the spreading capability of species 2 without competition enables it to survive the environment worsening speed. The numerical results, presented in Figure 2, indicate that species 1 eventually becomes extinct in the habitat and the species 2 persists by spreading to the right with spread speed $c_2^*(\infty) = 2.2$, confirming the result in Theorem 2.2.

We further consider an even smaller value of c, $c = 1.65$. Then, $c < \hat{c}^*(\infty)$, a scenario of Theorem 2.7. The numeric simulations (see Figure 3) confirm that the two competing species co-persist in a spreading pattern, and their respective asymptotical spreading speeds seem to be $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$, respectively.

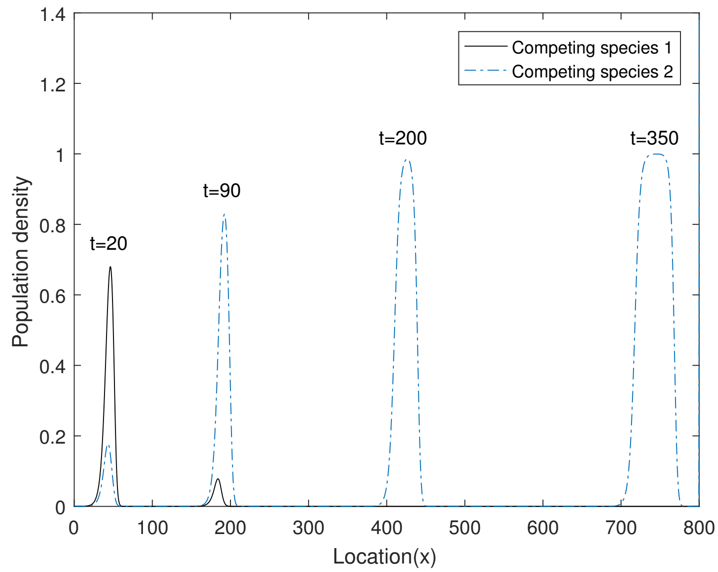

FIGURE 2. Numerical simulations on (6) with (113)-(116): when the environment worsening rate is neutral in the sense that $c = 2.05 \in (c_1^*(\infty), c_2^*(\infty))$, species 1 becomes extinct in the habitat and species 2 persist by spreading to the right with the asymptotic speed $c_2^*(\infty) = 2.2$.

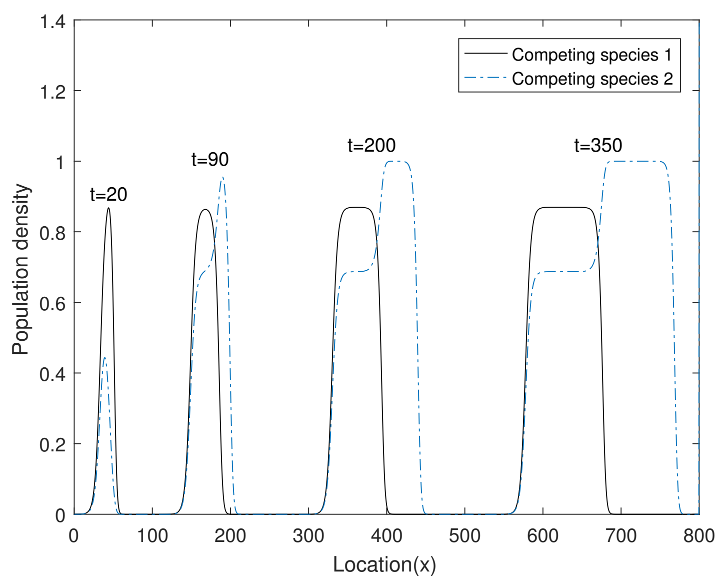

FIGURE 3. Numerical simulations on (6) with (113)-(116): when the environment worsening rate is very small ($c = 1.65$) in the sense that $c < \hat{c}^*(\infty)$, both species co-persist by spreading to the right with the respective asymptotic speeds $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$.

In Remark 2, we mentioned that $c < \hat{c}^*(\infty)$ is a sufficient condition but may not be necessary condition for the two species to co-persist by spreading to the right. We now demonstrate this by considering $c = 1.8$, which satisfies $c > \hat{c}^*(\infty)$ but $c < c^*(\infty)$. As is seen in the simulations presented in Figure 4, the two competing species can still co-persist by spreading to the right, and their asymptotic spread speeds still seem to be $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$, respectively.

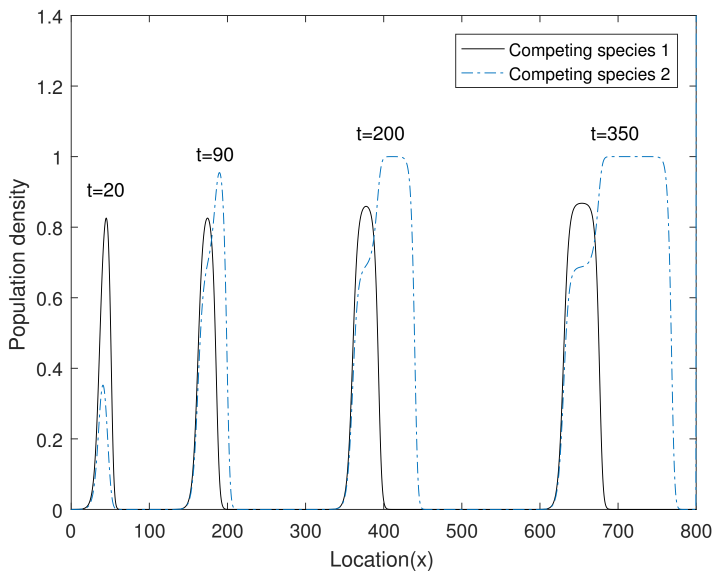

FIGURE 4. Numerical simulations on (6) with (113)-(116): for $c = 1.8$ we have $c > \hat{c}^*(\infty)$ but $c < c^*(\infty)$, the two species can still co-persist by spreading to the right with the respective asymptotic speeds $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$.

Next, we consider the case involving *strong competition*. For convenience, we still use $d_1 = 1$ and $d_2 = 1.21$ leading to $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$. Now we also take $c = 1.8$ as for Figure 4, but we replace the weak competition coefficients $a_1 = 0.19$ and $a_2 = 0.36$ by strong ones $a_1 = 3 > 1$ and $a_2 = 2 > 1$. The numerical simulations in Figure 5 show that, species 1 will go to extinction while species 2 persists — co-persistence is no longer the outcome. This confirms the conclusion of Theorem 2.11, and is in strong contrast to the results for weak competition illustrated in Figure 4 where co-persistence is observed.

In Remark 6, we raised the question of what happens if $c_2^* > c_1^* > c$ and $a_1 < 1 \leq a_2$, which we are unable to answer theoretically. Now we present some numerical results. We still use $d_1 = 1$ and $d_2 = 1.21$ leading to $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$ and choose $c = 1.8$ as in Figure 5, but choose $a_1 = 0.19$ and $a_2 = 3$. The simulation simulation result is given in Figure 6 which indicates that both species can co-persist by spreading to the right with the respective asymptotic speeds $c_1^*(\infty) = 2$ and $c_2^* = 2.2$.

4. **Discussion.** As mentioned in Remark 2, *under the weak competition* condition $a_i \in (0, 1)$, $i = 1, 2$, $c < \hat{c}^*(\infty)$ is a sufficient condition for the two species to co-persist by spreading toward the right. The numeric results presented in Section 3

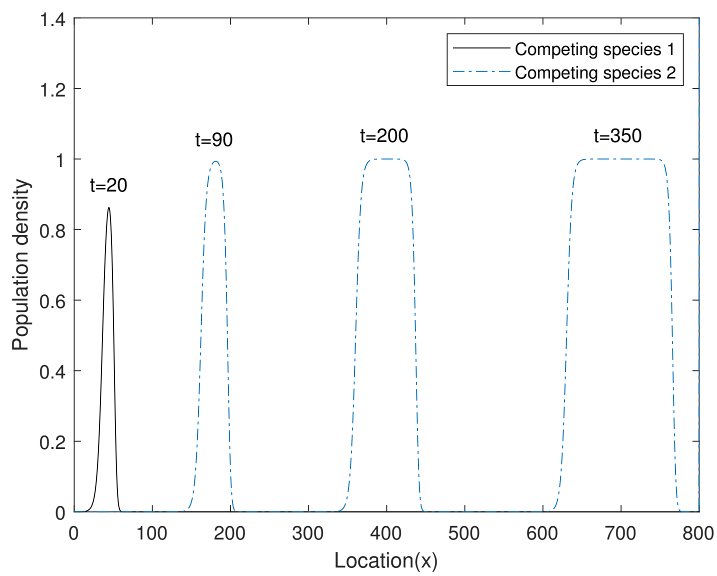

FIGURE 5. Numerical simulations on (6) with (113)-(116) with $a_1 = 0.19$ and $a_2 = 0.36$ replaced by $a_1 = 3$ and $a_2 = 2$ respectively. The environment worsening rate is very small (c=1.8) in the sense that $c < c_1^*(\infty)$, species 1 becomes extinct in the habitat and species 2 persist by spreading to the right with the asymptotic speed $c_2^*(\infty) = 2.2$.

seem to suggest that $c < c^*(\infty)$ is the *necessary and sufficient condition* for the two species to be able to co-persist by spreading to the right. Actually the theoretical results obtained in Section 2 and the simulation results presented in Section 3 seem to suggest that (i) $c_i^*(\infty) > c$ is the *necessary and sufficient condition* for the species i to be able to persist by spreading to the right (assuming reasonable initial distribution); and (ii) if $c_i^*(\infty) > c$ then the species i will persist by spreading to the right with the asymptotic spread speed $c_i^*(\infty)$ (see, Remark 1). If the above *conjectures* are true, then the weak competition does not affect the outcome and it is the individual intrinsic spreading capability compared to the environment worsening speed that determines the long time spatial dynamics of the species' population. We point out that when studying traveling wave fronts of autonomous Lotka-Velterra type diffusive cooperative/competitive systems that connect the extinction equilibrium and the co-existence equilibrium (assuming existence), the minimal wave speed is also independent of the competition strengths (see, e.g., [22, 43, 44]), and this offers another motivation to the above conjectures. Noticing that $c_i^*(\infty) = 2\sqrt{d_i r(\infty)}$ is increasing in d_i, the above discussion indicates that in such a situation (in whole space $\mathbb{R}$ with the environment worsening at a constant speed), evolution would favour a faster dispersal rate, *in strong contrast* to the situation of bounded domain with heterogeneous static environment in which slower dispersal is favoured. This is reasonable, because when there is no limit in space, *weak competition* is less relevant and spatial invasion/spread plays the dominant role for a species to survive.

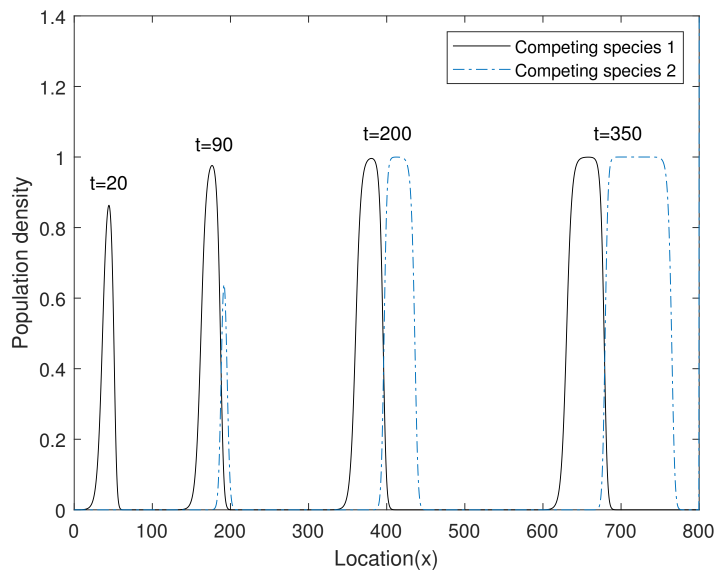

FIGURE 6. Numerical simulations on (6) with (113)-(116) with $a_2 = 0.36$ replaced by $a_2 = 3$ and the environment worsening rate is very small ($c = 1.8$) in the sense that $c < c_1^*(\infty)$, the two species can still be co-persist by spreading to the right with the respective asymptotic speeds $c_1^*(\infty) = 2$ and $c_2^*(\infty) = 2.2$.

In the case involving strong competition, situation would be different. Our results indicate that the interplay of the species' competing strengths and the spreading speeds also has an effect on the spatial dynamics. Theorem 2.11 shows that if the faster species is also a strong competitor ($a_i > 1$) and if its spreading speed is faster than the environment worsening speed, then slower species will go to extinction, regardless of whether it is a strong or weak competitor and whether it spreads faster or slower than the environment worsening speed. A ecological explanation for this given in Remark 6. An particular interesting scenario is that one species is a strong competitor while the other is a weak competitor (i.e., $((a_1 - 1)(a_2 - 1) < 0)$. Particularly, we find that a strong but slower competitor can co-persist with a weak but faster competitor, provided that the environment worsening speed is not too fast, as demonstrated in Figure 6. Such an outcome of co-persistence ought to be a result of balancing the capabilities of spreading and competition. Similar phenomenon is also observed in [13] where a system of form (5) (or (3) with constant r with $a_1 < 1 < a_2$ and $d_1 < d_2$ is considered. Particular attention of [13] is to the effect of initial functions.

Our results are obtained for *both cases of weak and strong competition*. The methods developed in [27] (also in subsection 2.1 of this paper) for weak competition and in subsection 2.2 of this paper for the case involving strong competitio do not seem to apply (at least directly) to the case of $c < c_1^*(\infty)$ and $0 < a_1 < 1 \leq a_2$. We have to leave the theoretical analysis of this case for a future work, using the numeric result in Figure 6 as an intuition, but developing some new method.

We point out that after we have written up our results into the first draft of this paper, Dr. Wendi Wang drew our attention (in an conference) to the publication of the most recent paper [49] by him and his co-authors (we thank Dr. Wang for this). In [49], the authors considered a system that is a little bit more general than (5) in the sense that the resource related growth functions for the two species can be different, that is,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r_1(x-ct) - u_1 - a_1 u_2], \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r_2(x-ct) - a_2 u_1 - u_2], \end{cases} \quad t>0, \;\; x \in \Omega \subset \mathbb{R}. \tag{117}$$

Here we would like to comment on some differences. Firstly, [49] is mainly motivated by [27], aiming to extend the work [27] to *competitive system* of Lotka-Volterra type. However, our work is motivated by those results on evolution of dispersals such as [9, 15, 17, 18, 19, 23, 24, 25, 31, 32] *as well as* [26], intending to compare the impact of different dispersal rates on the temporal-spatial dynamics in the Lotka-Volterra system when encountering a special *shifting habitat*. As such, we assume the same grow function for the two competing species (as in the most of above mentioned works) and focus on comparing the distinct dispersal rates and competition strengths. This setting enables us to obtain some more detailed results. For example, in the *weak competition case* ([49] *solely* deals with weak competition case), while the only theorem of [49] (Theorem 1) contains a result that is the corresponding version of our Theorem 2.7, it does not contain a results corresponding to Theorem 2.3. Consequently, [49] only established a lower bound $\hat{c}^*(\infty)$ for the spreading speed of the species i when $c \in (0, \hat{c}_i^*(\infty))$, however, because of Theorem 2.3, our results here give not only the lower bound but also an upper bound for the spreading speed $\hat{c}_i$ of the species i when $0 < c < \hat{c}_i^*(\infty)$ (see Remark 1). Moreover, for the case of $r_1(x) = r_2(x)$ the corresponding co-persistence portion of the results in [49] (i.e., Theorem 1-a), under the assumption $d_1 < d_2$, requires that the initial distributions for the two species satisfy

$$0 < \phi_1(x) < u_1^+, u_2^+ < \phi_2(x) \leq r_2(\infty)$$

on a closed interval, while our Theorem 2.7 only requires

$$0 < \phi_i(x) \leq r(\infty)$$

on a closed interval.

Appendix. In this appendix, we give the detailed proof of Lemma 2.5.

Proof of Lemma 2.5. Firstly, we consider the bounded case, i.e., the case in which $\mathbb{R}$ is replaced by a bounded interval $[-L, L]$, where $L > 0$. Therefore, the boundary value problem (12) can be written as

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} = d_1 \dfrac{\partial^2 v_1(t,x)}{\partial x^2} + H_1(t,x,v(t,x)), \;\; (t,x) \in Q, \\ \dfrac{\partial v_2(t,x)}{\partial t} = d_2 \dfrac{\partial^2 v_2(t,x)}{\partial x^2} + H_2(t,x,v(t,x)), \;\; (t,x) \in Q, \\ v(t,x) = \phi(t,x), \;\; (t,x) \in \Gamma, \end{cases} \tag{118}$$

where $\phi = (\Phi_1, \Phi_2)$, $\Phi_i = \max\{\min\{\widetilde{v}_i, \Psi_i\}, \widehat{v}_i\}$, Ψ_i is a continuous extension of θ_i in $\mathbb{R}_+ \times \mathbb{R}$, $i = 1, 2$, $Q = (0,T) \times (-L, L)$ and $\Gamma = (0,T) \times \{-L, L\} \bigcup \{0\} \times [-L, L]$. Obviously, $\phi(0,x) = \theta(x)$ for all $x \in \mathbb{R}$ and $\widehat{v} \leq \phi \leq \widetilde{v}$.

A vector function $v \equiv (v_1, v_2)$ is said to be a continuous weak upper (lower) solution of (118) if v is continuous on $\overline{Q}$, $v|_\Gamma \geq (\leq)\phi$ and

$$\frac{\partial v_i(t,x)}{\partial t} \geq (\leq) d_i \frac{\partial^2 v_i(t,x)}{\partial x^2} + H_i(t,x,v(t,x))$$

in the distributional sense, i.e., for any $\eta_i \in C^{1,2}([0,T]\times(-L,L))$ with $\eta_i \geq 0$ and $\text{supp}\eta_i(t,\cdot) \Subset (-L,L)$ for all $t \in [0,T]$,

$$\int_{-L}^{L} v_i(t,x)\eta_i(t,x)dx \Big|_{t=0}^{t=T_1}$$
$$\geq (\leq) \int_0^{T_1}\int_{-L}^{L} \left[v_i(s,x)\left(d_i\eta_{i,xx} + \eta_{i,t}\right)(s,x) + \eta_i(s,x)H_i(s,x,v(s,x))\right] dxds$$

if $T_1 \in [0,T]$, where $\eta_{i,xx}(s,x) = \frac{\partial^2\eta_i(t,x)}{\partial x^2}\Big|_{(t,x)=(s,x)}$ and $\eta_{i,t}(s,x) = \frac{\partial \eta_i(t,x)}{\partial t}\Big|_{(t,x)=(s,x)}$, $i = 1,2$. Let $\widetilde{v} \equiv (\widetilde{v_1}, \widetilde{v_2})$ and $\widehat{v} \equiv (\widehat{v_1}, \widehat{v_2})$ be the continuous weak upper and lower solutions of (118). Clearly, they are also the continuous weak upper and lower solutions of (12) as $L \to \infty$. Let

$$k_i = \max\left\{ \frac{\partial H_i(t,x,v)}{\partial v_i} \Bigg| (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \widehat{v} \leq v \leq \widetilde{v} \right\},$$

and

$$H_i^{**}(t,x,v) = k_i v_i + H_i(t,x,v),\ i = 1,2.$$

Then H_i^{**} is nondecreasing with respect to $v \in [\widehat{v}, \widetilde{v}]$ for $i = 1,2$.

Let $G_i(t,x,y)$ be Green's function of

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} = d_i \dfrac{\partial^2 v_i(t,x)}{\partial x^2} - k_i v_i(t,x),\ (t,x) \in Q, \\ v_i(t,x) = 0, \quad (t,x) \in \Gamma \end{cases}$$

and v_{i,Φ_i} the classical solution of

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} = d_i \dfrac{\partial^2 v_i(t,x)}{\partial x^2} - k_i v_i(t,x),\ (t,x) \in Q, \\ v_i(t,x) = \Phi_i(t,x), \quad (t,x) \in \Gamma, \end{cases}$$

where $i = 1,2$. G_i and v_{i,Φ_i} can be obtained by the Perron method (see [12]). Define $v_i^{(0)} \equiv \widehat{v}_i$, $v_i^{(1)} \equiv v_{i,\Phi_i} + T_i(v_i^{(0)})$, where

$$T_i(v_i^{(0)})(t,x) = \int_0^t \int_{-L}^{L} G_i(d_i(t-s),x,y)H_i^{**}(s,y,\widehat{v}(s,y))dyds,\ i = 1,2.$$

By the same arguments as in the proof of [46] (Lemma 1.2), we can obtain that $T_i(v_i^{(0)})$ is a continuous weak solution of

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} = d_i \dfrac{\partial^2 v_i(t,x)}{\partial x^2} - k_i v_i(t,x) + H_i^{**}(t,x,\widehat{v}(t,x)),\ (t,x) \in Q, \\ v_i(t,x) = 0, \quad (t,x) \in \Gamma. \end{cases} \tag{119}$$

Therefore, $v_i^{(1)} = v_{i,\Phi_i} + T_i(v_i^{(0)})$ is a continuous weak solution of

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} = d_i \dfrac{\partial^2 v_i(t,x)}{\partial x^2} - k_i v_i(t,x) + H_i^{**}(t,x,\widehat{v}(t,x)),\ (t,x) \in Q, \\ v_i(t,x) = \Phi_i(t,x), \quad (t,x) \in \Gamma, \end{cases} \tag{120}$$

that is, $v^{(1)} \equiv (v_1^{(1)}, v_2^{(1)})$ is a continuous weak solution of

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} = d_1 \dfrac{\partial^2 v_1(t,x)}{\partial x^2} - k_1 v_1(t,x) + H_1^{**}(t,x,\widehat{v}(t,x)),\ (t,x) \in Q, \\ \dfrac{\partial v_2(t,x)}{\partial t} = d_2 \dfrac{\partial^2 v_2(t,x)}{\partial x^2} - k_2 v_2(t,x) + H_2^{**}(t,x,\widehat{v}(t,x)),\ (t,x) \in Q, \\ v(t,x) = \phi(t,x),\ (t,x) \in \Gamma. \end{cases} \tag{121}$$

Observe $\widetilde{v}$ and $\widehat{v}$ are continuous weak upper and lower solutions of (118) (note $H_i^{**}(t,x,\widehat{v}) \le H_i^{**}(t,x,\widetilde{v})$, $i = 1,2$). By the strong maximum principle for weakly subparabolic functions, we have

$$\widehat{v} \le v^{(1)} \le \widetilde{v}$$

on $\overline{Q}$ (see [11]). Define $v_i^{(j)} = v_{i,\Phi_i} + T_i(v_i^{(j-1)})$, $i = 1,2$. Then similarly as above, we have

$$\widehat{v} \le v^{(1)} \le v^{(2)} \le \cdots \le v^{(j)} \le \widetilde{v}$$

on $\overline{Q}$, where $v^{(j)} \equiv (v_1^{(j)}, v_2^{(j)})$, $j = 1, 2, \cdots$. Let

$$v = \lim_{j\to\infty} v^{(j)}$$

and $v \equiv (v_1, v_2)$. Then by the Lebesgue Dominated Convergence Theorem,

$$\lim_{j\to\infty} T_i(v_i^{(j)}) = T_i(v_i), i = 1, 2,$$

and by the same reasoning regarding $v^{(1)}$ in the previous paragraph, we can prove that $v = (v_{1,\Phi_1} + T_1(v_1), v_{2,\Phi_2} + T_2(v_2))$ is a continuous weak solution of (118). Then a bootstrap argument implies that v is a classical solution of (118). Obviously, $\widehat{v} \le v \le \widetilde{v}$. Thus, we have completed the proof of Lemma 2.5 in case of bounded interval $[-L, L]$.

Next, we consider the case of bounded interval $(-\infty, \infty)$. Take an increasing sequence $L^{(j)} > 0$ such that $L^{(j)} \to \infty$ as $j \to \infty$. Let $Q^{(j)} = (0,T) \times (-L^{(j)}, L^{(j)})$ and $\Gamma^{(j)}$ be the parabolic boundary of $Q^{(j)}$. Then $Q^{(j)} \to (0,T) \times \mathbb{R}$ and $\Gamma^{(j)} \to \{0\} \times \mathbb{R}$ as $j \to \infty$. Consider the boundary value problem

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} = d_1 \dfrac{\partial^2 v_1(t,x)}{\partial x^2} + H_1(t,x,v(t,x)),\ (t,x) \in Q^{(j)}, \\ \dfrac{\partial v_2(t,x)}{\partial t} = d_2 \dfrac{\partial^2 v_2(t,x)}{\partial x^2} + H_2(t,x,v(t,x)),\ (t,x) \in Q^{(j)}, \\ v(t,x) = \phi(t,x),\ (t,x) \in \Gamma^{(j)}. \end{cases} \tag{122}$$

By the conclusion for bounded interval proved above, (122) has a classical solution $v^{(j)} \equiv (v_1^{(j)}, v_2^{(j)})$ with $\widehat{v} \le v^{(j)} \le \widetilde{v}$ for each $j \ge 1$. Then applying L^P interior estimates and Schauder interior estimates to $v^{(j)}$, we have

$$\|v_i^{(j)}\|_{C^{1+\alpha/2, 2+\alpha}(\Omega')} \le M(Q'), \quad \forall Q' \Subset (0,T) \times \mathbb{R},$$

where $M(Q')$ is independent of j, $0 < \alpha < 1$ and $i = 1,2$. From this and a diagonalization argument, there is a subsequence of $\{v^{(j)}\}_{j=1}^{j=\infty}$ (still denote it by $\{v^{(j)}\}_{j=1}^{j=\infty}$) such that $v_i^{(j)} \to v_i$ in $C_{loc}^{1,2}((0,T) \times \mathbb{R})$, where $i = 1, 2$. Obviously, v satisfies the differential equation in (12) and $\widehat{v} \le v \le \widetilde{v}$ on $(0,T) \times \mathbb{R}$. By the same arguments as in the proof of [46] (Lemma 1.2), we have $v_i \in C([0,T] \times \mathbb{R})$ and $v_i(0,x) = \theta_i(x)$ for all $x \in \mathbb{R}$, where $i = 1,2$. The proof of Lemma 2.5 is completed. □

Acknowledgments. The authors would like to thank the two anonymous referees for their critical and valuable comments which have led to a substantial and significant improvement in the revision, including the addition of Subsection 2.2 dealing with the case involving strong competition.

REFERENCES

[1] D. G. Aronson and H. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve impulse propagation, in *Partial Differential Equations and Related Topics, Lecture Notes in Math.,* Springer-Verlag, Berlin, **446** (1975), 5–49.

[2] H. Berestycki, O. Diekmann, C. J. Nagelkerke and P. A. Zegeling, Can a species keep pace with a shifting climate?, *Bull. Math. Biol.*, **71** (2009), 399–429.

[3] H. Berestycki and J. Fang, Forced waves of the Fisher-KPP equation in a shifting environment, *J. Differential Equations*, **264** (2018), 2157–2183.

[4] B. A. Bradley, D. S. Wilcove and M. Oppenheimer, Climate change increases risk of plant invasion in the eastern United States, *Biological Invasions*, **12** (2010), 1855–1872.

[5] R. S. Cantrell and C. Cosner, The effects of spatial heterogeneity in population dynamics, *J. Math. Biol.*, **29** (1991), 315–338.

[6] R. S. Cantrell and C. Cosner, On the effects of spatial heterogeneity on the persistence of interacting species, *J. Math. Biol.*, **37** (1998), 103–145.

[7] R. S. Cantrell and C. Cosner, *Spatial Ecology Via Reaction-Diffusion Equations*, J. Wiley, Chichester, 2003.

[8] C. Carrére, Spreading speeds for a two-species competition-diffusion system, *J. Differential Equations*, **264** (2018), 2133–2156.

[9] J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, The evolution of slow dispersal rates: A reaction-diffusion model, *J. Math. Biol.*, **37** (1998), 61–83.

[10] J. Fang, Y. Lou and J. Wu, Can pathogen spread keep pace with its host invasion?, *SIAM J. Appl. Math.* , **76** (2016), 1633–1657.

[11] A. Friedman, A strong maximum principle for weakly subparabolic functions, *Pacific J. Math.*, **11** (1961), 175–184.

[12] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[13] L. Girardin and K.-Y. Lam, Invasion of an empty habitat by two competitors: spreading properties of monostable two-species competition–diffusion systems, preprint, arXiv:1803.00454.

[14] P. Gonzalez, R. P. Neilson, J. M. Lenihan and R. J. Drapek, Global patterns in the vulnerability of ecosystems to vegetation shifts due to climate change, *Glob. Ecol. Biogeogr.*, **19** (2010), 755–768.

[15] A. Hastings, Can spatial variation alone lead to selection for dispersal? *Theor. Popul. Biol.*, **24** (1983), 244–251.

[16] A. Hastings, Spatial heterogeneity and ecological models, *Ecology*, **71** (1990), 426–428.

[17] X. He and W. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system I: Heterogeneity vs. homogeneity, *J. Differential Equations*, **254** (2013), 528–546.

[18] X. He and W. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system II: The general case, *J. Differential Equations*, **254** (2013), 4088–4108.

[19] X. He and W. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system: diffusion and spatial heterogeneity I, *Comm. Pure Appl. Math.*, **69** (2016), 981–1014.

[20] C. Hu and B. Li, Spatial dynamics for lattice differential equations with a shifting habitat, *J. Differential Equations*, **259** (2015), 1967–1989.

[21] H. Hu and X. Zou, Existence of an extinction wave in the Fisher equation with a shifting habitat, *Proc. Amer. Math. Soc.*, **145** (2017), 4763–4771.

[22] J. Huang and X. Zou, Traveling wavefronts in diffusive and cooperative Lotka-Volterra system with delays, *J. Math. Anal. Appl.*, **271** (2002), 455–466.

[23] V. Hutson, Y. Lou and K. Mischaikow, Spatial heterogeneity of resources versus Lotka-Volterra dynamics, *J. Differential Equations*, **185** (2002), 97–136.

[24] V. Hutson, Y. Lou and K. Mischaikow, Convergence in competition models with small diffusion coefficients, *J. Differential Equations*, **211** (2005), 135–161.

[25] V. Hutson, S. Martinez, K. Mischaikow and G. T. Vickers, The evolution of dispersal, *J. Math. Biol.*, **47** (2003), 483–517.

[26] S. A. Levin, Dispersion and population interactions, *Amer. Natur.*, **108** (1974), 207–228.
[27] B. Li, S. Bewick, J. Shang and W. F. Fagan, Persistence and spread of a species with a shifting habitat edge, *SIAM J. Appl. Math.*, **74** (2014), 1397–1417.
[28] G. Lin and W.-T. Li, Asymptotic spreading of competition diffusion systems: The role of interspecific competitions, *European J. Appl. Math.*, **23** (2012), 669–689.
[29] S. R. Loarie, P. B. Duffy, H. Hamilton, G. P. Asner, C. B. Field and D. D. Ackerly, The velocity of climate change, *Nature*, **462** (2009), 1052–1055.
[30] T. E. Lovejoy and L. Hannah, *Climate Change and Biodiversity*, Yale University Press, New Haven, 2005.
[31] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differential Equations*, **223** (2006), 400–426.
[32] Y. Lou, Some reaction diffusion models in spatial ecology (in Chinese), *Sci. Sin. Math.*, **45** (2015), 1619–1634.
[33] J. P. McCarty, Ecological consequences of recent climate change, *Conserv. Biol.*, **15** (2001), 320–331.
[34] A. Okubo and S. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, second ed., Interdisciplinary Applied Mathematics, vol. 14, Springer, New York, 2001.
[35] S. Pacala and J. Roughgarden, Spatial heterogeneity and interspecific competition, *Theor. Popul. Biol.*, **21** (1982), 92–113.
[36] C. V. Pao, Dynamics of nonlinear parabolic systems with time delays, *J. Math. Anal. Appl.*, **198** (1996), 751–779.
[37] C. Parmesan, N. Ryrholm, C. Stefanescu, J. K. Hill, C. D. Thomas, H. Descimon, B. Huntley, L. Kaila, J. Kullberg, T. Tammaru, W. J. Tennent, J. A Thomas and M. Warren, Poleward shifts in geographical ranges of butterfly species associated with regional warming, *Nature*, **399** (1999), 579–583.
[38] C. Parmesan, Ecological and evolutionary responses to recent climate change, *Ann. Rev. Ecology Evolution Systematics*, **37** (2006), 637–669.
[39] C. L. Parr, E. F. Gray and W. J. Bond, Cascading biodiversity and functional consequences of a global change-induced biome switch, *Divers. Distrib.*, **18** (2012), 493–503.
[40] A. B. Potapov and M. A. Lewis, Climate and competition: The effect of moving range boundaries on habitat invisibility, *Bull. Math. Biol.*, **66** (2004), 975–1008.
[41] B. Sandel, L. Arge, B. Dalsgaard, R. G. Davies, K. J. Gaston, W. J. Sutherland and J.-C. Svenning, The influence of late quaternary climate-change velocity on species endemism, *Science*, **334** (2011), 660–664.
[42] S. Scheiter and S. I. Higgins, Impacts of climate change on the vegetation of Africa: An adaptive dynamic vegetation modelling approach, *Glob. Change Biol.*, **15** (2009), 2224–2246.
[43] M. M. Tang and P. C. Fife, Propagation fronts in competing species equations with diffusion, *Arch. Rational Mech. Anal.*, **73** (1980), 69–77.
[44] J. H. van Vuuren, The existence of traveling plane waves in a general class of competition-diffusion systems, *IMA J. Appl. Math.*, **55** (1995), 135–148.
[45] G. R. Walther, E. Post, P. Convey, A. Menzel, C. Parmesan, T. J. C. Beebee, J. M. Fromentin, O. Hoegh-Guldberg and F. Bairlein, Ecological responses to recent climate change, *Nature*, **416** (2002), 389–395.
[46] X. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Amer. Math. Soc.*, **337** (1993), 549–590.
[47] H. F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.*, **13** (1982), 353–396.
[48] D. Xia, Z. Wu, S. Yan and W. Shu, *Theory of Real Variable Function and Functional Analysis*, second ed., Higher Education Press, Beijing, 2010. (in Chinese)
[49] Z. Zhang, W. Wang and J. Yang, Persistence versus extinction for two competing species under climate change, *Nonlinear Analysis: Modelling and Control*, **22** (2017), 285–302.
[50] Y. Zhou and M. Kot, Discrete-time growth-dispersal models with shifting species ranges, *Theor. Ecol.*, **4** (2011), 13–25.

Received for publication October 2018.

E-mail address: yuanyueding2005@163.com
E-mail address: ywan342@uwo.ca
E-mail address: xzou@uwo.ca