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Dynamics in Numerics: On a Discrete Predator-prey Model*

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Abstract

In this paper, we consider the dynamics of a discrete predator-prey model which is a result of discretization of the corresponding continuous Lotka-Volterra predator-prey model. Among the topis are equilibria and their stability, existence and stability of a period-2 orbit, as well the chaotic behavior of F. The chaos here is in the sense of topological horseshoe and is obtained for certain range of parameter values by applying a recent result from [5]. Our results are in contrast to the recent ones in [6] which claimed that if the predator-prey interaction is replaced by cooperative or competitive interaction, the discretization preserves the property of convergence to the equilibrium, regardless of the step size.

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1. Introduction

Most differential equation models in population biology (as well as in other areas) can not be solved explicitly, and numeric solutions are inevitable. In a recent work Huang and Zou [4], the authors examined the dynamics in two different numeric schemes for the scalar Kolmogorov type ordinary differential equations

$$\frac{dx}{dt} = xf(x). \tag{1.1}$$

One is the traditional Euler's scheme (a derivative approximation) and the other is based on an integral approximation (IA). By examining the equilibria and their stability, periodic orbits and their stability, and even topological chaos of these two schemes, we obtained some comparison results showing merits and drawbacks of each of these two schemes. However, due to interactions of species in real word, population models more than often are given by systems of ordinary differential equations. Taking the predator-prey type interaction as an example, a classic continuous time predator-prey model is

$$\begin{aligned} \tfrac{dx}{dt} &= x(t)[\mu_1 - \mu_1 x(t) - s_1 y(t)] \\ \tfrac{dy}{dt} &= y(t)[-\mu_2 + s_2 y(t)]. \end{aligned} \tag{1.2}$$

Let $t_n = n\,h$ where h is a step size. Applying the constant-variation-formula to each equation in (1.2), one obtains

$$\begin{aligned} x(t_{n+1}) &= x(t_n)\exp(\textstyle\int_{t_n}^{t_{n+1}}[\mu_1 - \mu_1 x(\xi) - s_1 y(\xi)]d\xi) \\ &\approx x(t_n)\exp(\mu_1 - \mu_1 x(t_n) - s_1 y(t_n)]h) \\ y(t_{n+1}) &= y(t_n)\exp(\textstyle\int_{t_n}^{t_{n+1}}[-\mu_2 + s_2 y(\xi)]d\xi) \\ &\approx y(t_n)\exp([-\mu_2 + s_2 y(t_n)]h). \end{aligned} \tag{1.3}$$

Thus, the following difference equation system gives a numeric scheme for (1.2):

$$x(t_{n+1}) = x(t_n)\exp(\mu_1 - \mu_1 x(t_n) - s_1 y(t_n)]h) \tag{1.4}$$
$$y(t_{n+1}) = y(t_n)\exp([-\mu_2 + s_2 y(t_n)]h).$$

The purpose of this paper is to explore the dynamic of this discretization of (1.2), as what we did in [4] to the two numeric schemes for (1.1).

Denoting $x(t_n) = x_n$ and $y(t_n) = y_n$, re-scaling $hs_1 y_n \to y_n$ and also re-scaling parameters $h\mu_1 \to \mu_1$ and $hs_2 \to h$, (1.4) becomes

$$x_{n+1} = x_n \exp[\mu_1 - \mu_1 x_n - y_n] \tag{1.5}$$
$$y_{n+1} = y_n \exp[-\mu_2 + s x_n],$$

where $s = s_2$. Obviously, all parameters in (1.5) are positive and due to the biological reason, we are only interested in $(x_n, y_n) \in \mathbb{R}^+ \times \mathbb{R}^+$. For convenience of discussion, we let $F = F_{\mu_1,\mu_2,s} : \mathbb{R}^+ \to \mathbb{R}^+$ be the right hand function in (1.4), that is,

$$F(x, y) = (x\exp[\mu_1 - \mu_1 x - y], y\exp[-\mu_2 + sx]). \tag{1.6}$$

In the rest of this paper, we only need to study (1.5), or the dynamics of the two dimensional map F defined by (1.6). In Section 2, we discuss the equilibria and their stability. In Sections 3, we investigate the existence and stability of period-2 orbit. Finally in Section 4, we explore the chaotic behavior of F: by applying a recent result from [5], we show that F^2 allows a topological horseshoe implying that F may have a chaotic behavior in certain range of parameter values.

It is interesting to note that if the starting continous system (1.2) is replaced by a Lotka-Volterra system with *cooperative* or *competitive* interaction, the same discretization leads to a discrete system that preserves the property of convergence to the equilibrium, regardless of the step size; see Liu and Elaydi [6]. Our results shows that the nature of interactions in a continuous system has a crucial impact on the dynamics of its discretization.

2. Equilibria and their stability

It is clear that the coordinate axes $x = 0$ and $y = 0$ are invariant under F. The exponential form of F assures that the forward trajectory of the system with respect to positive initial values $x_0 > 0,\ y_0 > 0$ remains in the positive quadrant of the plane for all time. Furthermore, using the inequality $e^u \geq 1 + u$, one can easily show that all trajectories are attracted to the stripe

$$D = \{(x, y) \mid 0 \leq x \leq \frac{1}{\mu_1} \exp(\mu_1 - 1),\ 0 \leq y < +\infty\},$$

in the first quadrant of the xy-plane.

For any $\mu_1 > 0, \mu_2 > 0$ and $s > 0$, F always has two fixed points

$$E_0 = (0, 0), \qquad \text{and} \qquad E_1 = (1, 0).$$

If $\mu_2 \geq s$, there is no other fixed point in $\mathbb{R}^+$; and if $\mu_2 < s$, then in addition to the the above two fixed points, F has a unique positive fixed point

$$E_+ = (\frac{\mu_2}{s},\ \mu_1(1 - \frac{\mu_2}{s})).$$

To do linearized stability analysis, we need some notions. A fixed point of F is called a source, saddle, or sink, respectively, if the Jacobian matrix of F at this fixed point has both eigenvalues with modulus greater than 1, one eigenvalue with modulus greater than 1 and the other less than 1, or both eigenvalues with modulus less than 1. The Jacobian matrix of F is calculated as below:

$$J(x, y) = DF(x, y) = \begin{pmatrix} (1 - \mu_1 x)e^{\mu_1 - \mu_1 x - y} & -ye^{\mu_1 - \mu_1 x - y} \\ sye^{-\mu_2 + sx} & e^{-\mu_2 + sx} \end{pmatrix}. \qquad (2.1)$$

The eigenvalues of $J(x, y)$ at the fixed point $(0, 0)$ are e^{μ_1} and $e^{-\mu_2}$. Therefore the origin is a saddle for all $\mu_1, \mu_2 > 0$. The local stable manifold theorem (see, for example [3]) assures that the local unstable manifold at the origin is tangent to the $x-$axis and the local stable manifold is tangent to the $y-$axis.

The local stability of the fixed point $(1, 0)$ depends on the parameters $\mu_1,\ \mu_2$ and s. The eigenvalues of $J(x, y)$ at this point are $\lambda_1 = 1 - \mu_1$

and $\lambda_2 = e^{-\mu_2+s}$. By the local stable and unstable manifold theorem, in a neighborhood of $(1, 0)$, there are two local invariant manifolds: one is tangent at $(1, 0)$ to the $x-$axis and is a stable manifold when $\mu_1 < 2$, but becomes unstable when $\mu_1 > 2$; the other intersects the $x-$axis transversely and is stable if $\mu_2 > s$ and becomes unstable if $\mu_2 < s$.

For the fixed point $(1, 0)$, in the parameter space

$$\mathbb{R}^3_+ = \{(\mu_1, \mu_2, s) \mid \mu_1,\ \mu_2,\ s > 0\},$$

except for the point on the line $\mu_1 = 2,\ \mu_2 = s$, along the boundary supper-surfaces $\mu_1 = 2$ or $\mu_2 = s$ there is only one eigenvalue of modulus 1. For the surface $\mu_1 = 2$, the eigenvalue of J is -1 and the fixed point $(1, 0)$ changes from stable to unstable, simultaneously with the occurrence of a two-cycle (see the next section). Thus, a flip bifurcation or periodic doubling occurs as (μ_1, μ_2, s) crosses the boundary surface $\mu_1 = 2$. As μ_1 further increases, periodic doubling continues and eventually chaotic behavior occurs as μ_1 increases up to a point of accumulation.

As the parameters (μ_1, μ_2, s) cross the plane $\mu_2 = s$ from the region $\mu_2 > s$ to the region $\mu_2 < s$, the bifurcation is quite different. When (μ_1, μ_2, s) crosses the surface $\mu_2 = s$, the fixed point $(1, 0)$ changes from stable to unstable and a new positive fixed point $(\frac{\mu_2}{s},\ \mu_1(1 - \frac{\mu_2}{s}))$ is generated.

Let

$$\begin{aligned}
D_1 &= \{(\mu_1, \mu_2, s) \in \mathbb{R}^3_+ \mid \mu_1 < 2,\ \mu_2 > s\},\\
D_2 &= \{(\mu_1, \mu_2, s) \in \mathbb{R}^3_+ \mid \mu_1 > 2,\ \mu_2 > s\},\\
D_3 &= \{(\mu_1, \mu_2, s) \in \mathbb{R}^3_+ \mid \mu_1 < 2,\ \mu_2 < s\},\\
D_4 &= \{(\mu_1, \mu_2, s) \in \mathbb{R}^3_+ \mid \mu_1 > 2,\ \mu_2 < s\}.
\end{aligned}$$

Then the following theorem summarizes the above results on the fixed point $E_1 = (1, 0)$.

Theorem 2.1. *The following results hold:*

(i) If $(\mu_1, \mu_2, s) \in D_1$, then E_1 is a sink;

(ii) If $(\mu_1, \mu_2, s) \in D_2 \cup D_3$, then E_1 is a saddle;

(iii) If $(\mu_1, \mu_2, s) \in D_4$, then E_1 is a source;

(iv) Flip bifurcation occurs when (μ_1, μ_2, s) *crosses the plane* $\mu_1 = 2$;

(v) Equilibrium bifurcation occurs when (μ_1, μ_2, s) *crosses the plane* $\mu_2 = s$.

Next, we study the local behavior of the positive fixed point $E_+ = (\mu_2/s, \mu_1(1-\mu_2/s))$ under the assumption $\mu_2 < s$. From (2.1), we can obtain the eigenvalue equation of the Jacobian at E_+ as below:

$$\lambda^2 - (2-\mu_1\alpha)\lambda + 1 + \mu_1\mu_2 - \mu_1\alpha(1+\mu_2) = 0, \tag{2.2}$$

where $\alpha \triangleq \mu_2/s < 1$. By Jury criterion (see, for example, [1], p. 82), the fixed point E_+ is stable if and only if the following two conditions hold simultaneously:

$$\begin{cases} \mu_1\mu_2 - \mu_1\alpha(1+\mu_2) < 0, \\ 2 + \mu_1\mu_2 - \mu_1\alpha(1+\mu_2) > |2-\mu_1\alpha|, \end{cases} \tag{2.3}$$

which is equivalent to

$$\begin{cases} \frac{\mu_2}{1+\mu_2} < \alpha < 1, \\ \\ \alpha < \frac{4+\mu_1\mu_2}{2\mu_1+\mu_1\mu_2}. \end{cases} \tag{2.4}$$

Hence, if

$$\frac{4+\mu_1\mu_2}{2\mu_1+\mu_1\mu_2} \leq \frac{\mu_2}{1+\mu_2}, \tag{2.5}$$

that is, if $\mu_1 \geq 4(1+\frac{1}{\mu_2})$, then E_+ is always unstable for any $s > \mu_2$. If $0 < \mu_1 < 4(1+\frac{1}{\mu_2})$, then E_+ is stable if and only if

$$\frac{\mu_2}{1+\mu_2} < \alpha < \min\left\{1,\ \frac{4+\mu_1\mu_2}{2\mu_1+\mu_1\mu_2}\right\}. \tag{2.6}$$

That is,

$$\mu_2 \max\{1, \frac{2\mu_1+\mu_1\mu_2}{4+\mu_1\mu_2}\} < s < 1+\mu_2. \tag{2.7}$$

By a more detailed analysis on the eigenvalue equation (2.2), we obtain the following classification results about the positive fixed point E_+.

Theorem 2.2. *Assume* $s > \mu_2$ *so that the positive fixed point* $E_+ = (\frac{\mu_2}{s},\ \mu_1(1 - \frac{\mu_2}{s}))$ *of* F *exists. Then*

(i) when $0 < \mu_1 < 2$, E_+ *is a sink for* $\mu_2 < s < 1 + \mu_2$ *but is a source for* $s > 1 + \mu_2$.

(ii) when $2 < \mu_1 < 4(1 + \frac{1}{\mu_2})$, *there are three possible cases:*

(ii-a) if

$$\mu_2 < s < \mu_1\mu_2\frac{2 + \mu_2}{4 + \mu_1\mu_2},$$

then E_+ *is a saddle;*

(ii-b) if

$$\mu_1\mu_2\frac{2 + \mu_2}{4 + \mu_1\mu_2} < s < 1 + \mu_2,$$

then E_+ *is a sink.*

(ii-c) if $s > 1 + \mu_2$, *then* E_+ *is a source.*

(iii) when $\mu_1 > 4(1+\frac{1}{\mu_2})$, E_+ *is a saddle if* $\mu_2 < s < 1+\mu_2$ *and a source if* $s > 1 + \mu_2$.

Stability diagram on the $\mu_2 - s$ plane for the the fixed point E_+ is given in Figure 1, where C represents the curve

$$s = \mu_1\mu_2\frac{2 + \mu_2}{4 + \mu_1\mu_2}, \qquad (2.8)$$

for a fixed μ_1.

We conclude this section by showing that if $s < \mu_2$ then the predator species will go to extinction.

Theorem 2.3. *If* $\mu_2 > s$, *then, for the system (1.2), we have* $\lim_{n\to\infty} y_n = 0$, *for any initial condition* $x_0 > 0,\ y_0 > 0$.

Proof. From (1.2), we have

$$\begin{aligned} x_{n+1}^s y_{n+1}^{\mu_1} &= x_n^s y^{\mu_1} \exp[\mu_1 s - \mu_1\mu_2 - y_n] \\ &\le x_n^s y^{\mu_1} \exp[-\mu_1(\mu_2 - s)]. \end{aligned}$$

Thus

$$x_{n+1}^{s}y_{n+1}^{\mu_1} \longrightarrow 0,$$

as $n \to \infty$. On the other hand, since $x_n \neq 0$ for all n and the local unstable manifold at the origin is tangent to the $x-$axis, there exist a constant $\delta > 0$ and a positive integer N such that $x_n > \delta$ for all $n > N$. Therefore one must have $\lim_{n\to\infty} y_n = 0$, completing the proof.

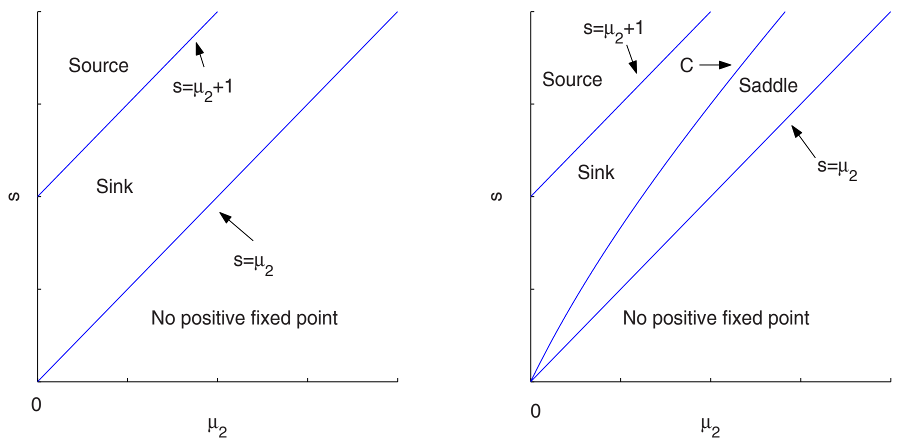

Figure 1: Stability diagram in $\mu_2 - s$ space for the positive fixed point E_+ for given $0 < \mu_1 < 2$ and $\mu_1 = 4 \in (2, 4(1 + \frac{1}{\mu_2}))$, respectively.

A diagram in terms of parameters μ_2 and s for the stability of E_+ is given in Figure 1.

3. Period two orbits

In this section, we are concerned with the existence of non-trivial period-2 obits of F and their stability. A non-trivial period-2 orbit of F corresponds to a period two point of F which is not a fixed point of F.

It is obvious that on the invariant line $x = 0$, F has no periodic point for any $\mu_1 > 0$, $\mu_2 > 0$ and $s > 0$. On the $x-$axis, which is also invariant under F, F is a one dimensional unimodal map, and has no non-trivial period-2 point if $\mu_1 < 2$. But when $\mu_1 > 2$, F has a period-2 orbit

$\{P_i = (x_i, 0),\ i = 1, 2\}$ with

$$x_i = 1 - \frac{2w_i}{\mu_1}, \qquad i = 1,\ 2, \tag{3.1}$$

where $w_1,\ w_2$ are the two nonzero roots of the equation

$$2w = \mu_1 \tanh(w). \tag{3.2}$$

Next we explore positive period-2 point of F. A direct calculation shows that

$$F^2(x, y) = (x \exp(h_1(x, y)), y \exp(h_2(x, y))), \tag{3.3}$$

where

$$h_1(x, y) = 2\mu_1 - \mu_1 x - y - \mu_1 x e^{\mu_1 - \mu_1 x - y} - y e^{-\mu_2 + sx}, \tag{3.4}$$
$$h_2(x, y) = -2\mu_2 + sx + sxe^{\mu_1 - \mu_1 x - y}. \tag{3.5}$$

Thus, positive period-2 points of F are governed by the following two equations:

$$2\mu_1 - \mu_1 x - y - \mu_1 x e^{\mu_1 - \mu_1 x - y} - y e^{-\mu_2 + sx} = 0,$$
$$-2\mu_2 + sx + sxe^{\mu_1 - \mu_1 x - y} = 0.$$

Solving these equations, we obtain

$$y = \frac{2\mu_1(s - \mu_2)}{s(1 + e^{-\mu_2 + sx})} \triangleq \psi(x), \tag{3.6}$$
$$y = \mu_1 + \ln x + \ln s - \ln(2\mu_2 - sx) - \mu_1 x \triangleq \phi(x). \tag{3.7}$$

A careful study of these two curves defined by (3.6) and (3.7) shows that a necessary condition for (3.6)-(3.7) to have a positive solution is

$$s > \mu_2, \qquad 0 < x < \frac{2\mu_2}{s}. \tag{3.8}$$

By a routine calculation, we can show that ψ possesses the following properties:

(i) For $s > \mu_2$, $\psi(x)$ is strictly decreasing on $x \in [0, \frac{2\mu_2}{s}]$;

(ii) $\psi(0) = \frac{2\mu_1(s-\mu_2)}{s(1+e^{-\mu_2})}$, $\psi(\frac{\mu_2}{s}) = \frac{\mu_1(s-\mu_2)}{s}$ and $\psi(\frac{2\mu_2}{s}) = \frac{2\mu_1(s-\mu_2)}{s(1+e^{\mu_2})}$.

(iii) $\psi'(\frac{\mu_2}{s}) = -\frac{1}{2}\mu_1(s-\mu_2)$.

From (3.7), we observe that

$$\lim_{x\to 0^+}\phi(x) = -\infty, \quad \lim_{x\to \frac{2\mu_2}{s}^-}\phi(x) = +\infty, \quad \text{and} \quad \phi(\frac{\mu_2}{s}) = \frac{\mu_1(s-\mu_2)}{s}.$$

By the above, one easily sees that $(\frac{\mu_2}{s}, \frac{\mu_1(s-\mu_2)}{s})$ is positive solution of (3.6)-(3.7), but obviously it is precisely the positive fixed point E_+ of F under the condition that $s > \mu_2$ as we have seen in Section 2 (so, it is not a true period-2 point of F).

A direct calculation gives

$$\phi'(x) = \frac{1}{x(2\mu_2 - sx)}(\mu_1 sx^2 - 2\mu_1\mu_2 x + 2\mu_2). \tag{3.9}$$

By the knowledge on quadratic functions, we know that when $0 < \mu_1 < 2$ (noting that $s > \mu_2$), $\phi'(x) > 0$ for $x \in (0, \frac{2\mu_2}{s})$. This, together with the properties of $\psi(x)$, shows that there is only one solution for (3.6)-(3.7) which corresponds to the positive fixed point of F. Therefore, in this case, F has no nontrivial period two point.

When $\mu_1 > 2$, $\phi'(x)$ may change sign on $(0, \frac{2\mu_2}{s})$ the two curves $y = \psi(x)$ and $y = \phi(x)$ may have either exactly one intercept which gives the fixed point $(\frac{\mu_2}{s}, \frac{\mu_1(s-\mu_2)}{s})$ for F, or three intercepts. The latter occurs if and only if

$$\psi'(\frac{\mu_2}{s}) > \phi'(\frac{\mu_2}{s}), \tag{3.10}$$

that is

$$-\frac{1}{2}\mu_1(s-\mu_2) > \frac{1}{\mu_2}(2s - \mu_1\mu_2).$$

The above inequality holds if and only if

$$s < \frac{\mu_1\mu_2(\mu_2+2)}{4+\mu_1\mu_2}.$$

In the later case, the two intercepts forms an positive period-2 orbit. Therefore, we have obtained the following results on existence of nontrivial period-2 orbit.

Theorem 3.1. *The following conclusions hold:*

(i) If $0 < \mu_1 < 2$, *then for any* $s > 0$ *and* $\mu_2 > 0$, F *has no non-trivial period-2 orbit.*

(ii) If $2 < \mu_1$ *and* $s > \frac{\mu_1\mu_2(\mu_2+2)}{4+\mu_1\mu_2}$, *then* F *has a non-trivial period-2 orbit on* $x-$*axis and there is no other non-trivial period-2 orbit.*

(iii) If $2 < \mu_1$ *and* $\mu_2 < s < \frac{\mu_1\mu_2(\mu_2+2)}{4+\mu_1\mu_2}$, *then, in addition to the boundary period-2 orbit stated in (ii), there is a unique positive non-trivial period-2 orbit for* F.

Remark 3.1. *From Theorems 2.2 and 3.1, in the parameter space* (μ_1, μ_2, s), *for* $\mu_1 > 2$, *both Hopf bifurcation and flip bifurcation occur when either* s *crosses the surface* $s = \mu_2$ *from below to above, or* s *crosses the surface* $s = \frac{\mu_1\mu_2(\mu_2+2)}{4+\mu_1\mu_2}$ *from above to below. Figure 2 illustrates the regions in the* $\mu_2 - s$ *plane corresponding to the existence and non-existence of the positive period two orbit.*

When $\mu_1 > 2$ and $\mu_2 < s < \frac{\mu_1\mu_2(\mu_2+2)}{4+\mu_1\mu_2}$, denote by $\{P_i = (x_i, y_i),\ i = 1, 2\}$ the corresponding positive non-trivial period-2 orbit. By the discussion above, we know that $0 < x_1 < \frac{\mu_2}{s}$ and $\frac{\mu_2}{s} < x_2 < \frac{2\mu_2}{s}$. In the rest of this section, we will study stability of this non-trivial period-2 orbit.

When $\mu_1 > 2$, the stable dynamics for the nontrivial period two point on $x-$axis is the same as that for one dimensional exponential type map. Here we only discuss the stable dynamics for the positive nontrivial period two point P_1 and P_2. Local asymptotic behavior of F at the the period two point can be described by that of F^2 at its fixed point P_1. A direct calculation shows that

The Jacobian matrix of F^2 at P_1 is

$$\begin{pmatrix} 1 + x_1 h'_{1x}(x_1, y_1) & x_1 h'_{1y}(x_1, y_1) \\ y_1 h'_{2x}(x_1, y_1) & 1 + y_1 h'_{2y}(x_1, y_1) \end{pmatrix}.$$

The characteristic equation of F^2 at P_1 can be written as

$$\lambda^2 - (2 + a)\lambda + (1 + a + b) = 0, \tag{3.11}$$

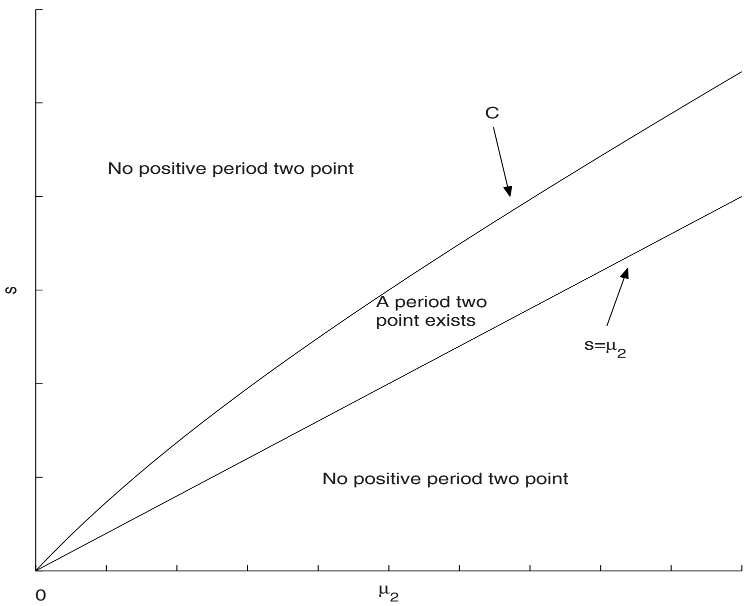

Figure 2: The regions in $a-b$ space corresponding to the existence and non-existence of the period-2 orbit, where $C: s = \frac{\mu_1\mu_2(\mu_2+2)}{4+\mu_1\mu_2}$ for $\mu_1 = 4$.

where

$$a = x_1 h'_{1x}(x_1, y_1) + y_1 h'_{2y}(x_1, y_1), \tag{3.12}$$

$$b = x_1 y_1 \left[h'_{1x}(x_1, y_1) h'_{2y}(x_1, y_1) - h'_{1y}(x_1, y_1) h'_{2x}(x_1, y_1)\right], \tag{3.13}$$

where x_1 is the solution of

$$\frac{2\mu_1(s-\mu_2)}{s(1+e^{-\mu_2+sx})} = \mu_1 + \ln x + \ln s - \ln(2\mu_2 - sx) - \mu_1 x \tag{3.14}$$

with

$$x \in \left(0, \frac{\mu_2}{s}\right) \quad \text{and} \quad y_1 = \frac{2\mu_1(s-\mu_2)}{s(1+e^{-\mu_2+sx_1})}. \tag{3.15}$$

In order to locate the roots of (3.14), we divide the $(a,b)-$plane into the following sub-regions:

$$
\begin{aligned}
I &= \{(a,b) \mid -4 < a < 0, \ \ \max\{0, \ -2a-4\} < b < -a\}, \\
II &= \{(a,b) \mid \max\{-a, \ \frac{a^2}{4}\} < b\}, \\
III &= \{(a,b) \mid a > 0, \ 0 < b < \frac{a^2}{4}\}, \\
IV &= \{(a,b) \mid a > -2, \ -2a-4 < b < 0\}, \\
V &= \{(a,b) \mid b < \min\{0, \ -2a-4\}\}, \\
VI &= \{(a,b) \mid a < -2, \ \ 0 < b < -2a-4\}, \\
VII &= \{(a,b) \mid a < -4, \ \ -2a-4 < b < \frac{a^2}{4}\}.
\end{aligned}
$$

By elementary analysis and some lengthy but straightforward calculations, we obtain the following

Theorem 3.2. *Let $\lambda_1, \ \lambda_2$ be the roots of the equation (3.14). Then we have*

(i) If $(a,b) \in I$, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$;

(ii) If $(a,b) \in II$, then $\lambda_1, \ \lambda_2$ are complex with $|\lambda_1| = |\lambda_2| > 1$;

(iii) If $(a,b) \in III$, then $\lambda_1 > 1$ and $\lambda_2 > 1$;

(iv) If $(a,b) \in IV$, then $-1 < \lambda_1 < 1$ and $\lambda_2 > 1$;

(v) If $(a,b) \in V$, then $\lambda_1 < -1$ and $\lambda_2 > 1$;

(vi) If $(a,b) \in VI$, then $\lambda_1 < -1$ and $-1 < \lambda_2 < 1$;

(vii) If $(a,b) \in VII$, then $\lambda_1 < -1$ and $\lambda_2 < -1$.

From this theorem, we know that only when $(a,b) \in I$, the positive period-2 orbit is stable. Theorem 3.2 is demonstrated in Figure 3 demonstrates the stability and instability regions in the $a-b$ plane for the positive non-trivial period-2 orbit. .

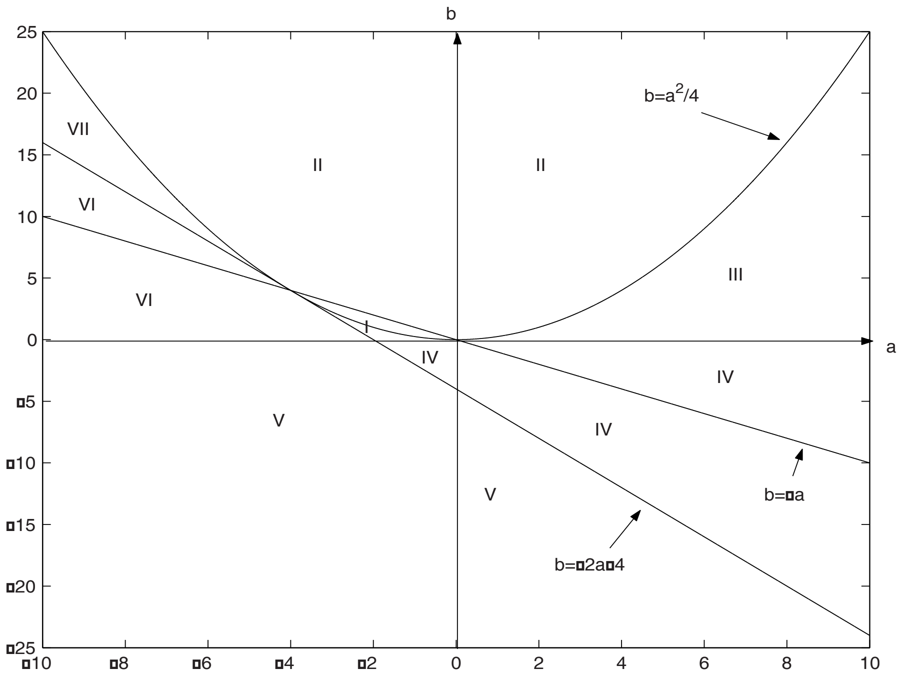

Figure 3: The stable and unstable regions of the positive nontrivial period-2 orbit of F.

4. Existence of topological horseshoe

In this section, we will prove the existence of topological horseshoe for F under some conditions. Roughly speaking, a dynamical system with topological horseshoe is semi-conjugate to a shift map. Since the latter has very complicated dynamical behavior such as chaos (see, e.g., [7],) a dynamical system with topological horseshoe also demonstrate chaotic behavior.

From Theorem 2.3, we know that the prey goes extinct if $\mu_2 > s$. We will show that even in such a case, system (2.2) may have a topological horseshoe. To this end, we appeal to a recent result on the existence of topological horseshoe from Kennedy and Yorke [5]. Let X be a separable metric space (here in our case $X = \mathbb{R}^2$), $Q \subset X$ be locally connected and compact subset and f be a continuous map from Q into X.

Lemma 4.1. *(***Topological horseshoe lemma,** [5] *) Let f be a continuous map from Q into X. Assume the following conditions hold:*

(A1) There exist two subsets of Q, denoted by Q_1 and Q_2, which are disjoint and compact.

(A2) Each connected component of Q intersects both Q_1 and Q_2.

(A3) The cross number m of Q with respect to f is not less than 2.

Then there exists a closed invariant set $Q_I \subset Q$ such that $f\,|_{Q_I}$ is semi-conjugate to a $m-$shift map, that is, there exists a continuous and onto map

$$h: \ Q_I \to \Sigma_m,$$

such that $h \circ f = \sigma \circ f$, where Σ_m is the conventional $m-$symbol space and σ is the shift map on σ.

Here the cross number m is defined as follows: A connection Γ for Q_1 and Q_2 is a connected compact subset of Q intersecting both Q_1 and Q_2. A pre-connection γ is a connected compact subset of Q such that $f(\gamma)$ is a connection. The cross number m is now defined as the largest number such that every connection contains at least m mutually disjoint pre-connection.

It turns out that it is very difficult, if not impossible, to prove the existence of topological horseshoe for F. Accidentally and luckily, we are able to prove that for F^2 and details follow below.

Set

$$\begin{aligned} h(x) &= xe^{\mu_1-\mu_1 x}, &(4.1)\\ g_1(x,y) &= \mu_1 h(x)(1-e^{-y}) - y(1+e^{-\mu_2+sx}), &(4.2)\\ g_2(x,y) &= -2\mu_2 + s(x+h(x)e^{-y}). &(4.3)\end{aligned}$$

Then by a direct calculation, F^2 can be written as

$$F^2(x,y) = (h^2(x)e^{g_1(x,y)},\ ye^{g_2(x,y)}). \tag{4.4}$$

The function $h(x)$ is of unimodal type, and this type of maps have been studied extensively. See, for example, [2]. The map $h(x)$ has two fixed points: 0 and 1. It is strictly increasing in $[0, \frac{1}{\mu_1}]$ and strictly decreasing in $[\frac{1}{\mu_1}, +\infty)$, and satisfies

$$\lim_{x\to+\infty} h(x) = 0.$$

Thus $h(x)$ has the global maximum value at $x = \frac{1}{\mu_1}$ and

$$M \triangleq h(\frac{1}{\mu_1}) = \frac{1}{\mu_1} e^{\mu_1 - 1}.$$

Therefore, when $\mu_1 > 1$, there exist r_1 and r_2 with $0 < r_1 < 1/\mu_1 < 1 < r_2$ such that

$$h^2(r_1) = h^2(r_2) = h(\frac{1}{\mu_1}) = M. \tag{4.5}$$

See Figure 4 for a demonstration of this fact.

The following lemma will be needed in proving the main result in this section.

Lemma 4.2. *There exists a constant μ_1^0 such that when $\mu_1 > \mu_1^0$, we have*

$$h^3(\frac{1}{\mu_1}) < \frac{1}{\mu_1}. \tag{4.6}$$

and

$$h^2(\frac{1}{\mu_1}) < r_1. \tag{4.7}$$

Proof. A routine calculation shows that

$$\begin{aligned} h^2(\frac{1}{\mu_1}) &= \frac{1}{\mu_1} \exp(2\mu_1 - 1 - e^{\mu_1 - 1}), \\ h^3(\frac{1}{\mu_1}) &= \frac{1}{\mu_1} \exp(3\mu_1 - 1 - e^{\mu_1 - 1} - e^{2\mu_1 - 1 - e^{\mu_1 - 1}}). \end{aligned}$$

Inequality (4.6) holds if and only if

$$3\mu_1 - 1 - e^{\mu_1 - 1} - e^{2\mu_1 - 1 - e^{\mu_1 - 1}} < 0. \tag{4.8}$$

A sufficient condition for (4.8) is

$$3\mu_1 - 1 - e^{\mu_1 - 1} < 0.$$

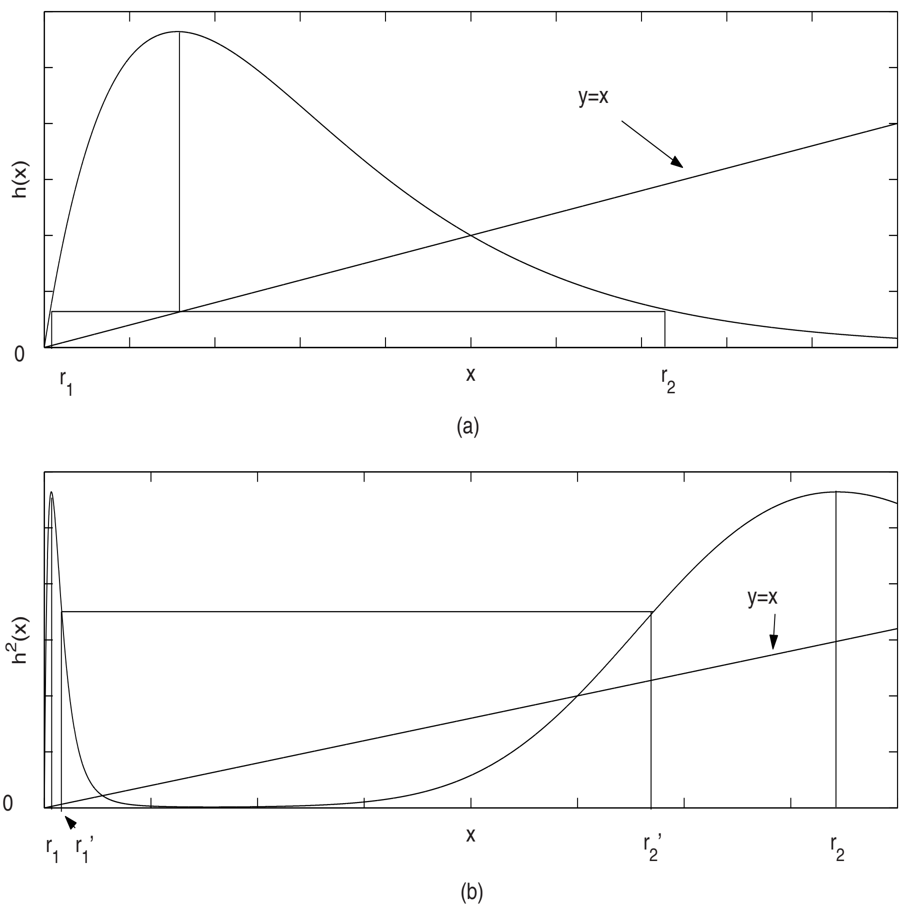

Figure 4: The graphs of the functions $h(x)$ and $h^2(x)$ with $\mu_1 = 3.2$.

This is obviously true for large enough μ_1. The last inequality also implies that

$$h^2(\frac{1}{\mu_1}) < \frac{1}{\mu_1}.$$

Thus, we have (4.7), since $h(\cdot)$ is increasing in $[0, \frac{1}{\mu_1}]$.

Remark 4.1. *Numerical computation shows that (4.6) holds when* $\mu_1 > 3.117$.

In the following, we fix $\mu_1 > \mu_1^0$. Since

$$h^2(r_1) = M > r_1, \qquad h^2(\frac{1}{\mu_1}) < \frac{1}{\mu_1},$$

there exists a fixed point of h^2 in the interval $(r_1, \frac{1}{\mu_1})$ which is a periodic-2 point of h. Denote by p_2 this periodic-2 point. From (4.5) and the unimodal properties of $h(x)$, it follows that for any $1 < r_2' < r_2$, there exists a unique r_1' with $r_1 < r_1' < p_2$ such that

$$h^2(r_1') = h^2(r_2') > r_2'. \tag{4.9}$$

For any $\varepsilon > 0$, let

$$Q_\varepsilon \triangleq [r_1', r_2'] \times [0, \varepsilon], \tag{4.10}$$

and

$$M_1 = \max_{r_1 < x < r_2} \{x + h(x)\}. \tag{4.11}$$

Theorem 4.1. *Let $\mu_1 > \mu_1^0$, where μ_1^0 is given by Lemma 4.2. If*

$$0 < s < \frac{2\mu_2}{M_1}, \tag{4.12}$$

then for any r_2' with $1 < r_2' < r_2$, there exists an $\varepsilon > 0$, such that the map $F^2: Q_\varepsilon \to \mathbb{R}^2$ has a topological horseshoe with cross number $m = 2$.

Proof. Let r_1' and r_2' satisfy (4.9). Since $\mu_1 > \mu_1^0$, by (4.7), there exist r_3, r_4 such that

$$p_2 < r_3 < \frac{1}{\mu_1} < r_4 < 1, \tag{4.13}$$

$$h^2(r_3) = h^2(r_4) < r_1 < r_1'. \tag{4.14}$$

On the other hand, we have from (4.2)

$$g_1(x, 0) = 0,$$

for any $x > 0$. It follows from (4.9) and (4.14) that there exists a $\varepsilon > 0$ such that

$$h^2(r_1')e^{g_1(r_1',y)} > r_2', \qquad h^2(r_2')e^{g_1(r_2',y)} > r_2', \tag{4.15}$$

$$h^2(r_3)e^{g_1(r_3,y)} < r_1', \qquad h^2(r_4)e^{g_1(r_4,y)} < r_1', \tag{4.16}$$

for any $y \in [0, \varepsilon]$. From (4.3) and (4.12), one can derive

$$ye^{g_2(x,y)} < y, \quad \text{for} \quad y > 0, \quad r_1 < x < r_2. \tag{4.17}$$

Now let Q_ε be defined by (4.10) and set

$$\begin{aligned} Q_1 &= \{(x,y) \mid x = r_1', \quad 0 \le y \le \varepsilon\}, \\ Q_2 &= \{(x,y) \mid x = r_2', \quad 0 \le y \le \varepsilon\}, \\ D_1 &= \{(x,y) \mid r_1' \le x \le r_3, \quad 0 \le y \le \varepsilon\}, \\ D_2 &= \{(x,y) \mid r_4 \le x \le r_2', \quad 0 \le y \le \varepsilon\}. \end{aligned}$$

If we denote $(\bar{x}, \bar{y}) = F^2(x,y)$, then by (4.15)-(4.17), for any $(\bar{x}, \bar{y}) \in$

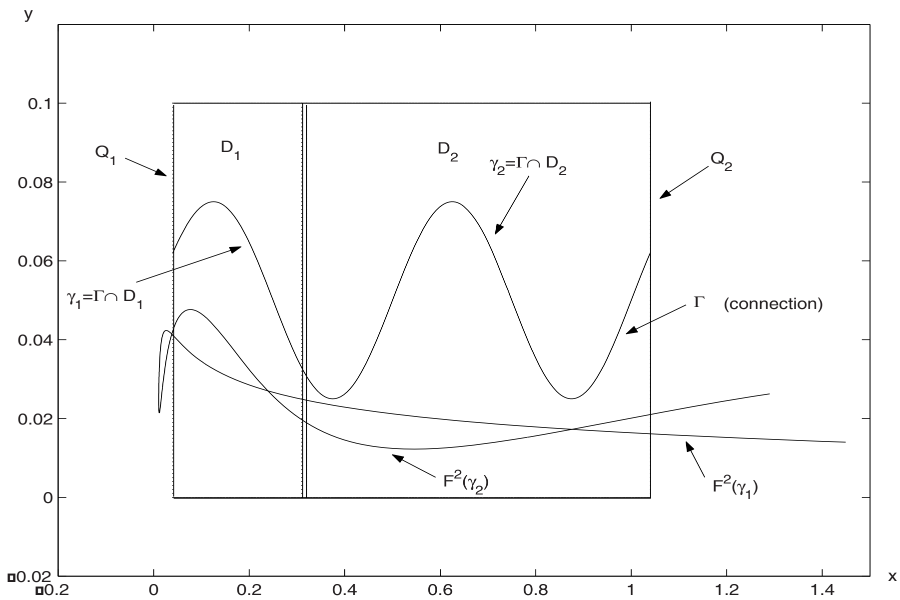

Figure 5: The existence of topological horseshoe for F^2 with $\mu_1 = 3.2$, $\mu_2 = 1, \quad s = 0.6$ and $\varepsilon = 0.1$.

$F^2(Q_i),\ i = 1, 2$, we have $\bar{x} > r_2'$ and $0 < \bar{y} < \varepsilon$. By the same argument, for any $(\bar{x}, \bar{y}) \in F^2(\{r_3\} \times [0, \varepsilon]),\ i = 1, 2$, we have $\bar{x} < r_1'$ and $0 < \bar{y} < \varepsilon$. Thus, if Γ is a connection, then by our discussion above, we can see that $\gamma_i \triangleq \Gamma \cap D_i,\ i = 1,\ 2$, are two mutually disjoint preconnections, since the curve $F^2(\gamma_i)$, $i = 1, 2$, cross Q_1 and Q_2 (see Figure 5 for a illustration). Hence, the three conditions (A1)-(A3) in Lemma 4.1 are verified for $f = F^2$ with $m = 2$, and therefore by this lemma, there exists a topological horseshoe for F^2. The proof is complete.

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