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Comparison Theorems Concerning the Oscillation of Higher Order Functional Differential Equations with Middle Terms

By

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1. Introduction

Research on the oscillation theory of functional differential equations of higher order has been intensive and extensive. Among the existing results, criteria for oscillation of the solutions of a given equation have been obtained usually by two ways: one is by the equation itself, and the other is by comparing with some other equations. In order to find suitable equations to be compared with, it is necessary to establish some comparison theorems. Kartsatos and Kosmala have done much on this aspect (see, e.g. [1]–[5]). But, to the best of the author's knowledge, the existing comparison results have been for the higher order equations with at most two middle terms (see [1]–[5]). As for the equations with more than two middle terms, the methods developed in [1]–[5] fail because the signs of the derivatives of a non-oscillatory solution in such cases are not necessarily definite (in fact, they cannot be determined for general case). The purpose of this paper is to develop a new technique which is more normal and universal to extend the related comparison theorems in [1]–[5] to more general equations of higher order, i.e. equations with all the middle terms. Even for the special cases of this paper, our results improve the related ones in [1]–[5] to a large extent (see the remarks at the end of this paper).

2. Preliminaries

Consider the equations

$$x^{(n)}(t) + \sum_{i=1}^{n} p_i(t)x^{(n-i)}(t) + H(t, x(g(t))) = 0 \tag{1}$$

$$x^{(n)}(t) + \sum_{i=1}^{n} p_i(t)x^{(n-i)}(t) + H(t, x(g(t))) = Q(t) \tag{2}$$

and

$$x^{(n)}(t)+\sum_{i=1}^{n}p_i(t)x^{(n-i)}(t)+H_1(t,x(g_1(t)))=Q(t) \tag{3}$$

$$x^{(n)}(t)+\sum_{i=1}^{n}p_i(t)x^{(n-i)}(t)+H_2(t,x(g_2(t)))=Q(t) \tag{4}$$

where $n \geqslant 4$ is even and

(A_1) $p_i \in C(\boldsymbol{R}_+, \boldsymbol{R})$ $i=1, 2, \ldots, n$

(A_2) $g, g_i \in C(\boldsymbol{R}_+, \boldsymbol{R})$, $\lim_{t\to\infty} g(t)=\infty$, $\lim_{t\to\infty} g_i(t)=\infty$ $i=1, 2$.

(A_3) for $i=1, 2$, H, $H_i \in C(\boldsymbol{R}_+, \boldsymbol{R}, \boldsymbol{R})$; $uH(t,u)>0$, $uH_i(t,u)>0$ for $u \neq 0$; $H(t,u)$ and $H_i(t,u)$ are nondecreasing in u.

By a solution of (1) ((2) or (3) or (4)) we mean any function $x \in C^n(t_x, \infty)$, which satisfies (1) ((2) or (3) or (4)) for all $t \in [t_x, \infty)$. Here t_x depends on the solution $x(t)$. As is customary, a solution of (1) ((2) or (3) or (4)) is said to be oscillatory if it has an unbounded set of zeros in its interval of definition $[t_x, \infty)$. If every solution of (1) ((2) or (3) or (4)) is oscillatory, then Eq. (1) ((2), (3) or (4)) is said to be oscillatory.

Let L be the differential operator defined as below

$$Lx(t)=x^{(n)}(t)+\sum_{i=1}^{n}p_i(t)x^{(n-i)}(t) \tag{5}$$

In what follows we are going to find another representation for L under the following condition:

(C) The ordinary differential equation $Lx(t)=0$ has fundamental solutions $x_1(t), x_2(t), \ldots, x_n(t)$ satisfying

$$w_r(t)>0 \qquad \text{for } t \geqslant t_0,\ r=1, 2, \ldots, n \tag{6}$$

where

$$w_r(t)=\begin{vmatrix} x_1(t) & x_2(t) & \vdots & x_r(t) \\ x_1'(t) & x_2'(t) & \vdots & x_r'(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(r-1)}(t) & x_2^{(r-1)}(t) & & x_r^{(r-1)}(t) \end{vmatrix} \qquad r=1, 2, \ldots, n \tag{7}$$

Lemma 1. *Suppose* (C) *holds, then L can be represented as*

$$Lx(t)=\frac{w_n(t)}{w_{n-1}(t)}\frac{d}{dt}\left(\frac{w_{n-1}^2(t)}{w_n(t)w_{n-2}(t)}\frac{d}{dt}\left(\cdots\frac{d}{dt}\left(\frac{w_1^2(t)}{w_2(t)w_0(t)}\frac{d}{dt}\left(\frac{x(t)}{w_1(t)}\right)\right)\cdots\right)\right) \tag{8}$$

where $w_r(t)$ is defined by (7) *for $r=1, 2, \ldots, n$ and $w_0(t)=1$.*

This is the Lemma 1 in [6]. For the sake of completeness we give the proof below.

Proof. Set $u_0(x(t)) = x(t)$ and

(9) $$u_r(x(t)) = \begin{vmatrix} x_1(t) & x_2(t) & \vdots & x_r(t) & x(t) \\ x_1'(t) & x_2'(t) & \vdots & x_r'(t) & x'(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{(r)}(t) & x_2^{(r)}(t) & & x_r^{(r)}(t) & x^{(r)}(t) \end{vmatrix} \qquad r = 1, 2, \dots, n$$

Since $w_n(t) \neq 0$ for $t \in [t_0, \infty)$, we conclude that

(10) $$u_n(x(t))/w_n(t) = 0$$

is a linear homogeneous ordinary differential equation of order n. Clearly $x_1(t), x_2(t), \dots, x_n(t)$ are also fundamental solutions of Eq. (10), and the coefficients of $x^{(n)}(t)$ both in Eq. (10) and equation $Lx(t) = 0$ are 1. According to the theory of ordinary differential equations we know that

(11) $$Lx(t) = u_n(x(t))/w_n(t)$$

Following the same line as in getting (11), we can easily obtain

(12) $$u_r(x(t))w_{r-1}(t) = u_{r-1}'(x(t))w_r(t) - u_{r-1}(x(t))w_r'(t) \qquad r = 1, 2, \dots, n.$$

and therefore

(13) $$\frac{u_r(x(t))}{w_r(t)} = \frac{u_{r-1}'(x(t))w_r(t) - u_{r-1}(x(t))w_r'(t)}{w_r(t)w_{r-1}(t)}$$

$$= \frac{w_r(t)}{w_{r-1}(t)} \frac{d}{dt}\left(\frac{u_{r-1}(x(t))}{w_r(t)}\right) \qquad r = 1, 2, \dots, n.$$

From (11) and (13), we obtain

(14) $$Lx(t) = \frac{w_n(t)}{w_{n-1}(t)} \frac{d}{dt}\left(\frac{u_{n-1}(x(t))}{w_n(t)}\right)$$

$$= \frac{w_n(t)}{w_{n-1}(t)} \frac{d}{dt}\left(\frac{w_{n-1}(t)}{w_n(t)} \frac{w_{n-1}(t)}{w_{n-2}(t)} \frac{d}{dt}\left(\frac{u_{n-2}(x(t))}{w_{n-1}(t)}\right)\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \frac{w_n(t)}{w_{n-1}(t)} \frac{d}{dt}\left(\frac{w_{n-1}^2(t)}{w_n(t)w_{n-2}(t)} \frac{d}{dt}\left(\cdots \frac{d}{dt}\left(\frac{w_1^2(t)}{w_2(t)w_0(t)} \frac{d}{dt}\left(\frac{u_0(x(t))}{w_1(t)}\right)\right)\cdots\right)\right)$$

Note that $u_0(x(t)) = x(t)$, we finally get (8). The proof is then complete.

In the sequel, we always assume the condition (C) holds. For the convenience of notations we set

$$(15)\qquad \begin{aligned} &a_0(t) = 1/x_1(t) \\ &a_i(t) = w_i^2(t)/w_{i+1}(t)w_{i-1}(t) \qquad \text{for } i = 1, 2, \ldots, n-1 \\ &a_n(t) = w_n(t)/w_{n-1}(t) \end{aligned}$$

and denote

$$(16)\qquad \begin{aligned} &L_0 x(t) = a_0(t)x(t) \\ &L_i x(t) = a_i(t)\frac{d}{dt}(L_{i-1}x(t)) \qquad \text{for } i = 1, 2, \ldots, n\,. \end{aligned}$$

Then, from Lemma 1 we conclude that $L = L_n$ and thus, Eq. (1)–(4) are equivalent respectively to

$$(1)' \qquad L_n x(t) + H(t, x(g(t))) = 0$$

$$(2)' \qquad L_n x(t) + H(t, x(g(t))) = Q(t)$$

$$(3)' \qquad L_n x(t) + H_1(t, x(g_1(t))) = Q(t)$$

$$(4)' \qquad L_n x(t) + H_2(t, x(g_2(t))) = Q(t)$$

It is obvious that $a_i(t)$ is continuous and $a_i(t) > 0$ for $t \in [t_0, \infty)$, $i = 0, 1, \ldots, n$. By virtue of [7], we can, through out this paper, always assume that

$$(17)\qquad \int^{\infty} 1/a_i(t)dt = \infty \qquad i = 1, 2, \ldots, n-1\,.$$

Lemma 2 ([8], Lemma 2). *Assume that* $x \in D(L_n)$, *and* $x(t)L_n x(t) < 0$ *eventually. Then, there exist* $t_x > 0$ *and odd number* k, $1 \leqslant k \leqslant n-1$, *such that for* $t \geqslant t_x$

$$\begin{aligned} x(t)L_i x(t) > 0 \qquad & i = 0, 1, \ldots, k \\ (-1)^j x(t)L_j x(t) < 0 \qquad & j = k+1, k+2, \ldots, n\,. \end{aligned}$$

3. Main results

In this section we establish four comparison theorems concerning Eq. (1)–(4).

Theorem 1. *Suppose* $s\colon [T, \infty)\, \boldsymbol{R}_+ \to \boldsymbol{R}$ *is continuous and* $x \in C^n[T, \infty), \boldsymbol{R})$.

(i) *If* $x(t) > 0$, $x(t) + s(t) > 0$ *and*

$$(18)\qquad L_n x(t) + H(t, x(g(t)) + s(g(t))) \leqslant 0$$

then, the equation

$$L_n y(t) + H(t, y(g(t)) + s(g(t))) = 0 \tag{19}$$

has an eventually positive solution $w(t)$.

(ii) *If* $x(t) < 0$, $x(t) - s(t) < 0$ *and*

$$L_n x(t) + H(t, x(g(t)) - s(g(t))) \geqslant 0 \tag{20}$$

then, the equation

$$L_n y(t) + H(t, y(g(t)) - s(g(t))) = 0 \tag{21}$$

has an eventually negative solution $v(t)$.

Proof. Since (ii) can be obtained from (i) simply by letting $u(t) = -x(t)$, we need only to prove (i).

By (A_3) and Lemma 2 and the conditions of this theorem, we know that there exist $t_1 \geqslant T$ and odd integer k, $1 \leqslant k \leqslant n-1$ such that for $t \geqslant t_1$

$$\begin{aligned} L_i x(t) > 0 \qquad & i = 0, 1, \ldots, k \\ (-1)^j L_j x(t) < 0 \qquad & j = k+1, \ldots, n \end{aligned} \tag{22}$$

As $n-1$ is odd, we know from (22) that $L_{n-1}x(t) > 0$ for $t \geqslant t_1$. Integrating (18) from t to u $(t_1 \leqslant t \leqslant u)$ yields

$$L_{n-1}x(u) - L_{n-1}x(t) + \int_t^u \frac{1}{a_n(r)} H(r, x(g(r)) + s(g(r)))dr \leqslant 0 \tag{23}$$

Noting that $L_{n-1}x(u) > 0$, we then have

$$L_{n-1}x(t) \geqslant \int_t^u \frac{1}{a_n(r)} H(r, x(g(r)) + s(g(r)))dr \tag{24}$$

Letting $u \to \infty$ in the above inequality, we have

$$L_{n-1}x(t) \geqslant \int_t^\infty \frac{1}{a_n(r)} H(r, x(g(r)) + s(g(r)))dr \tag{25}$$

Similarly, integrating the inequality (25) $n-k-1$ times and using (22), we obtain

$$\begin{aligned} L_k x(t) \geqslant & \int_t^\infty \frac{1}{a_{k+1}(r_{n-k-1})} \int_{r_{n-k-1}}^\infty \frac{1}{a_{k+2}(r_{n-k-2})} \cdots \int_{r_2}^\infty \frac{1}{a_{n-1}(r_1)} \int_{r_1}^\infty \frac{}{a_n(r)} \\ & \cdot H(r, x(g(r)) + s(g(r)))dr dr_1 \ldots dr_{n-k-1} \\ & \triangleq G(t, x) \end{aligned} \tag{26}$$

Now we integrate the inequality (26) k times from t_1 to $t \geqslant t_1$ and use (22) to obtain

$$(27)\qquad L_0x(t) \leqq L_0x(t_1) + \int_{t_1}^{t} \frac{1}{a_1(s_k)} \int_{t_1}^{s_k} \cdots \int_{t_1}^{s_2} \frac{1}{a_k(s_1)} G(s_1, x)ds_1 \dots ds_k$$
$$\triangleq L_0x(t_1) + \bar{G}(t, x)$$

Set

$$(28)\qquad w_0(t) = x(t) \qquad t \geqslant t_1$$
$$w_{m+1}(t) = \frac{1}{a_0(t)}[L_0x(t_1) + \bar{G}(t, w_m)] \qquad t \geqslant t_1 \text{ and } m = 0, 1, 2, \dots$$

By induction we know that the sequence $\{w_m(t)\}$ has the following properties:

$$(29)\qquad 0 < w_m(t) \leqslant x(t) \qquad t \geqslant t_1$$
$$x(t_1) \leqslant w_{m+1}(t) \leqslant w_m(t) \qquad t \geqslant t_1 \text{ and } m = 0, 1, 2, \dots$$

Consequently, we can apply Lebesgue's Monotone Convergence Theorem to $\{w_m(t)\}$ to conclude that there exists a function $w(t) \geqslant x(t_1) > 0$ such that $\lim_{m \to \infty} w_m(t) = w(t)$.

Letting $m \to \infty$ in (28) we get

$$(30)\qquad w(t) = \frac{1}{a_0(t)}[L_0x(t_1) + \bar{G}(t, w)] \qquad t \geqslant t_1$$

Noting the definition of $\bar{G}$ and differentiating (30) k times, we have

$$(31)\qquad L_k w(t) = G(t, w) \qquad t \geqslant t_1$$

By the definition of G, and again, differentiating (31) $n - k$ times, we get

$$(32)\qquad L_n w(t) = (-1)^{n-k} H(t, w(g(t)) + s(g(t))) \qquad t \geqslant t_1$$

Since $n - k$ is odd, (32) leads to

$$L_n w(t) + H(t, w(g(t)) + s(g(t))) = 0 \qquad t \geqslant t_1$$

which means that Eq. (19) has eventually positive solution $w(t)$. The proof of theorem 1 is then complete.

Remark 1. From the proof of (i) part of theorem 1, we can easily see that the eventually positive solution $w(t)$ of Eq. (19) satisfies $L_i w(t) > 0$, $i = 0, 1, \dots, k$ and $(-1)^j L_j w(t) < 0$, $j = k + 1, \dots, n$. The eventually negative solution of Eq. (21) in (ii) part of this theorem has also a similar property.

Theorem 2. *Assume that* $s\colon \boldsymbol{R}_+ \to \boldsymbol{R}$ *satisfies* $L_n s(t) = Q(t)$ *on* $\boldsymbol{R}_+$ *and is such that* $L_0 s(t) \to 0$ *as* $t \to \infty$. *If Eq.* (2) *is oscillatory, then, so is Eq.* (1).

Proof. For the sake of contradiction, we assume that Eq. (1) has a non-oscillatory solution $x(t)$. Without loss of generality, we can assume that $x(t)$ is eventually positive, i.e. there exists $t_1 > 0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for all $t \geqslant t_1$. By (1)′ and (A_3), we thus have

$$L_n x(t) = -H(t, x(g(t)) < 0 \qquad \text{for } t \geqslant t_1 \tag{33}$$

In the light of Lemma 2, we know that

$$\text{(34)} \quad L_0 x(t) > 0 \quad \text{and} \quad \frac{d}{dt}[L_0 x(t)] = \frac{1}{a_1(t)} L_1 x(t) > 0 \qquad \text{for } t \geqq t_1$$

Because $L_0 s(t) \to 0$ as $t \to \infty$, there exists $t_2 \geqslant t_1$ for given positive number $\varepsilon < \frac{1}{2} L_0 x(t_1)$ such that

$$\text{(35)} \qquad |L_0 s(t)| < \varepsilon \qquad \text{for } t \geqslant t_2$$

In terms of (34), we have

$$\text{(36)} \qquad L_0 x(t) \geqslant L_0 x(t_2) \geqslant L_0 x(t_1) > 2\varepsilon \qquad \text{for } t \geqslant t_2$$

We note from (A_2) that exists $t_3 \geqslant t_2$ such that $g(t) \geqslant t_2$ for $t \geqslant t_3$. So, if we set $u(t) = x(t) + s(t)$, then

$$\begin{aligned} \text{(37)} \qquad u(g(t)) - s(g(t)) &= \frac{1}{a_0(g(t))}[L_0 u(g(t)) - L_0 s(g(t))] \\ &> \frac{1}{a_0(g(t))}[L_0 u(g(t)) - \varepsilon] \\ &> \frac{1}{a_0(g(t))}[L_0 x(g(t)) - 2\varepsilon] > 0 \qquad \text{for } t \geqslant t_3 \end{aligned}$$

Adding $L_n s(t) = Q(t)$ to (1)′, we get

$$\text{(38)} \qquad L_n u(t) + H(t, u(g(t)) - s(g(t))) = Q(t)$$

Now we combine (37) and (A_3) with (38) to obtain

$$\text{(39)} \qquad L_n u(t) + H\left(t, u(g(t)) - \frac{\varepsilon}{a_0(g(t))}\right) < Q(t) \qquad \text{for } t \geqslant t_3$$

On the other hand, we clearly have $L_n\left(\frac{\varepsilon}{a_0(t)}\right) = 0$. We now set $v(t) = u(t) - s(t) - \frac{\varepsilon}{a_0(t)}$, then

$$\text{(40)} \qquad L_n v(t) + H(t, v(g(t)) + s(g(t))) < 0 \qquad \text{for } t \geqslant t_3$$

where

$$v(t) = u(t) - s(t) - \frac{\varepsilon}{a_0(t)} \tag{41}$$

$$= x(t) - \frac{\varepsilon}{a_0(t)}$$

$$= \frac{1}{a_0(t)}[L_0 x(t) - \varepsilon] > 0 \qquad \text{for } t \geqslant t_3$$

and

$$v(t) + s(t) = u(t) - \frac{\varepsilon}{a_0(t)} \tag{42}$$

$$= \frac{1}{a_0(t)}[L_0 u(t) - \varepsilon]$$

$$> \frac{1}{a_0(t)}[L_0 x(t) - 2\varepsilon] > 0 \qquad \text{for } t \geqslant t_3$$

By theorem 1 and remark 1, we know that there exist eventually positive function $w(t)$ and odd integer k, $1 \leqslant k \leqslant n-1$, such that

$$L_n w(t) + H(t, w(g(t)) + s(g(t))) = 0 \tag{43}$$

and

$$L_i w(t) > 0 \qquad i = 0, 1, \ldots, k\,. \tag{44}$$

Adding $L_n s(t) = Q(t)$ to (43), we get

$$L_n[w(t) + s(t)] + H(t, w(g(t)) + s(g(t))) = Q(t) \tag{45}$$

In vertue of (44) and $\lim_{t\to\infty} L_0 s(t) = 0$, we can easily conclude that $w(t) + s(t)$ is eventually positive. So, Eq. (2) has an eventually positive solution $w(t) + s(t)$. This is a contradiction to the assumption that Eq. (2) is oscillatory. The proof is then complete.

The next theorem is a reverse of theorem 2.

Theorem 3. *Assume that the conditions in theorem* 2 *hold. If* $s(t)$ *in the conditions is oscillatory, then, the oscillation of Eq.* (1) *implies the oscillation of Eq.* (2).

Proof. For the sake of contradiction, we assume that Eq. (2) is not oscillatory. Without loss of generality, we assume that Eq. (2) has an eventually positive solution $x(t)$. By (A_2), we see that there exists $t_1 > 0$ such that

$x(t) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_1$. Set $u(t) = x(t) - s(t)$ and subtract $L_n s(t) = Q(t)$ from Eq. (2), then we get

$$\begin{aligned} L_n u(t) &= -H(t, u(g(t)) + s(g(t))) \\ &= -H(t, x(g(t))) < 0 \qquad \text{for } t \geqslant t_1 \end{aligned} \tag{46}$$

Using (17) and reviewing the proof of Lemma 2, we can similarly conclude that there exist $t_2 \geqslant t_1$ and odd integer k, $1 \leqslant k \leqslant n-1$, such that for $t \geqslant t_2$

$$\begin{aligned} L_i u(t) &> 0 \qquad i = 1, 2, \ldots, k \\ (-1)^j L_j u(t) &< 0 \qquad j = k+1, \ldots, n \end{aligned} \tag{47}$$

Therefore,

$$\frac{d}{dt}[L_0 u(t)] = \frac{1}{a_1(t)} L_1 u(t) > 0 \qquad \text{for } t \geqslant t_2 \tag{48}$$

which implies that $\lim_{t\to\infty} L_0 u(t) = u_0$ exists.

We consider the following two cases:

Case 1. $u_0 \leqslant 0$. From (48), we would have $L_0 u(t) < 0$, and thus, $0 < L_0 x(t) = L_0 u(t) + L_0 s(t) < L_0 s(t)$ which is a contradiction to the assumption that $s(t)$ is oscillatory.

Case 2. $u_0 > 0$. There would be, in this case, $t_3 \geqslant t_2$ such that $u_0 > L_0 u(t) > L_0 u(t_3) > 0$ for $t \geqslant t_3$. Let ε be a positive number such that $\varepsilon < L_0 u(t_3)$. Since $\lim_{t\to\infty} L_0 s(t) = 0$ and $\lim_{t\to\infty} g(t) = \infty$, there would exist $t_4 \geqslant t_3$ such that $g(t) \geqslant t_3$ and $|L_0 s(g(t))| < \varepsilon$ for $t \geqslant t_4$. Consequently

$$\begin{aligned} u(g(t)) + s(g(t)) &= \frac{1}{a_0(g(t))}[L_0 u(g(t)) + L_0 s(g(t))] \\ &> \frac{1}{a_0(g(t))}[L_0 u(g(t)) - \varepsilon] \\ &= u(g(t)) - \frac{\varepsilon}{a_0(g(t))} \\ &\geqslant \frac{1}{a_0(g(t))}[L_0 u(t_3) - \varepsilon] > 0 \qquad \text{for } t \geqslant t_4 \end{aligned} \tag{49}$$

Combining (49) and (A_3) with (46), we obtain

$$L_n u(t) + H\left(t, u(g(t)) - \frac{\varepsilon}{a_0(g(t))}\right) \leqslant 0 \qquad \text{for } t \geqslant t_4 \tag{50}$$

Set $v(t) = u(t) - \frac{\varepsilon}{a_0(t)}$. As $L_n \frac{\varepsilon}{a_0(t)} = 0$, it follows from (50) that

$$L_n v(t) + H(t, v(g(t))) \leqslant 0 \qquad \text{for } t \geqslant t_4 \tag{51}$$

where $v(t) = u(t) - \frac{\varepsilon}{a_0(t)} = \frac{1}{a_0(t)}[L_0 u(t) - \varepsilon] > \frac{1}{a_0(t)}[L_0 u(t_3) - \varepsilon] > 0$ for $t \geqslant t_4$. It turns out from theorem 1 that Eq. (1) would have an eventually positive solution which contradicts the assumption that Eq. (1) is oscillatory.

The proof is completed.

Corollary. Let the hypotheses of theorem 3 hold. Then, Eq. (1) oscillates if and only if Eq. (2) oscillates.

The following is a comparison theorem concerning Eq. (3) and Eq. (4).

Theorem 4. *Let the hypotheses of theorem 3 hold. Furthermore, in addition to* (A_2) *and* (A_3), *we assume* g_i *and* H_i $(i = 1, 2)$ *satisfy*

(A_4) $g_1(t) \leqslant g_2(t)$ $t \geqslant 0$

(A_5) $u(H_2(t, u) - H_1(t, u)) \geqslant 0$ *for* $u \neq 0$

and $a_0(t)$ *is nonincreasing. Then, if Eq.* (3) *is oscillatory, then so is Eq.* (4).

Proof. Assume, for the sake of contradiction, that Eq. (4) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume $x(t)$ is eventually positive, i.e. there exists $t_1 > 0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_1$. Set $u(t) = x(t) - s(t)$, then

$$L_n u(t) + H_2(t, u(g_2(t)) + s(g_2(t))) = 0 \tag{52}$$

Since $u(g_2(t)) + s(g_2(t)) = x(g_2(t)) > 0$ for $t \geqslant t_1$, by (52) and (A_5) we get

$$L_n u(t) + H_1(t, u(g_2(t)) + s(g_2(t))) \leqslant 0 \qquad \text{for } t \geqslant t_1 \tag{53}$$

Therefore

$$L_n u(t) \leqq -H_1(t, u(g_2(t)) + s(g_2(t))) < 0 \qquad \text{for } t \geqslant t_1 \tag{54}$$

From the above inequality, we claim that (as in getting (48) in the proof of theorem 3) there exists $t_2 \geqslant t_1$ such that

$$\frac{d}{dt}[L_0 u(t)] = \frac{1}{a_1(t)} L_1 u(t) > 0 \qquad \text{for } t \geqslant t_2 \tag{55}$$

and thus $\lim_{t\to\infty} L_0 u(t) = u_0$ exists. The case $u_0 \leqslant 0$ is clearly a contradiction (by the same argument as in the proof of theorem 3), we need only to consider the case $u_0 > 0$.

From $u_0 > 0$ and (55) we know that there exist $t_3 \geqslant t_2$ such that $u_0 >$

$L_0u(t) \geqslant L_0u(t_3) > 0$ for $t \geqslant t_3$. Let ε be a positive number such that $\varepsilon < L_0u(t_3)$. Since $\lim_{t\to\infty} L_0s(t) = 0$ and $\lim_{t\to\infty} g_1(t) = \infty$, there would exist $t_4 \geqslant t_3$ such that $g_1(t) \geqslant t_3$ and $|L_0s(g_1(t))| < \varepsilon$ for $t \geqslant t_4$. Therefore

$$(56)\qquad \begin{aligned} u(g_2(t)) + s(g_2(t)) &= \frac{1}{a_0(g_2(t))}[L_0u(g_2(t)) + L_0s(g_2(t))] \\ &> \frac{1}{a_0(g_2(t))}[L_0u(g_2(t)) - \varepsilon] \\ &= u(g_2(t)) - \frac{\varepsilon}{a_0(g_2(t))} \\ &> \frac{1}{a_0(g_2(t))}[L_0u(t_3) - \varepsilon] > 0 \qquad \text{for } t \geqq t_4 \end{aligned}$$

In the light of (56), (A_3) and (53) we obtain

$$(57)\qquad L_nu(t) + H_1\left(t, u(g_2(t)) - \frac{\varepsilon}{a_0(g_2(t))}\right) < 0 \qquad \text{for } t \geqslant t_4$$

Observe that

$$(58)\qquad \begin{aligned} u(g_2(t)) - \frac{\varepsilon}{a_0(g_2(t))} &= \frac{1}{a_0(g_2(t))}[L_0u(g_2(t)) - \varepsilon] \\ &\geqslant \frac{1}{a_0(g_1(t))}[L_0u(g_2(t)) - \varepsilon] \\ &\geqslant \frac{1}{a_0(g_1(t))}[L_0u(g_1(t)) - \varepsilon] > 0 \qquad \text{for } t \geqslant t_4 \end{aligned}$$

Let $v(t) = u(t) - \dfrac{\varepsilon}{a_0(t)}$, then it follows that

$$(59)\qquad L_nv(t) + H_1(t, v(g_1(t))) < 0 \qquad \text{for } t \geqslant t_4$$

By theorem 1 for the case $s(t) = 0$, we conclude that there exists a positive function $w(t)$ on $[t_4, \infty)$ such that

$$(60)\qquad L_nw(t) + H_1(t, w(g_1(t))) = 0$$

It turns out (by theorem 2) that Eq. (3) would be non-oscillatory. This is a contradiction. The proof is then completed.

Remark 2. In the above theorems, we do not require that $g(t) \leqslant t$ which has been assumed in [1]–[5]. This means that our results are applicable to the equations of not only retarded type, but also advanced and even mixed type.

Remark 3. For the special case $p_i(t) \equiv 0$ for $i = 1, 2, \ldots, n$ and $g_1(t) \equiv g_2(t)$, theorem 4 leads to ([1], theorem 2.1) and ([2], theorem for n even), and theorem 3 leads to ([1], theorem 3.4).

Remark 4. For the case $p_i(t) \equiv 0$ for $i = 2, 3, \ldots, n$ and $p_1(t) \not\equiv 0$, our theorem 3 and theorem 2 lead to ([3], theorem 4.1 and corollary 4.2) and ([4], theorem 3.2 and theorem 3.6) respectively. We note that in [3] and [4], it has been required that $p_1 \in C[t_0, \infty)$ and

$$\lim_{t\to\infty} \int_{t_0}^{t} \exp\left[-\int_{t_0}^{u} p_1(s)ds\right]du = \infty \tag{61}$$

But here in this paper, in order that the condition (C) be satisfied we need only to assume $p_1 \in C[t_0, \infty)$ (see [6]). So the assumption (61) in [3] and [4] is indeed unnecessary. Besides, our theorem 1 implies that the condition $\lim_{t\to\infty} s(t) = 0$ in ([4], theorem 3.2) can also be removed.

Remark 5. For the case $p_i(t) \equiv 0$ for $i = 3, 4, \ldots, n$, Eq. (1)–Eq. (4) reduces to the equations studied in [5] and our results (theorem 1, corollary and theorem 4) lead to ([5], theorem 4.1, theorem 4.2 and theorem 4.4) respectively. Note that in [5], $p_1(t)$ and $p_2(t)$ have been assumed to satisfy the following conditions:

(i) $p_1(t) \leqslant 0$, $p_2(t) \geqslant 0$, $p_1(t)$ is differentiable and $p_2(t) - p_1'(t) \leqslant 0$

(ii) For any given $t_0 \geqslant 0$, $k > 0$ and $M \in \boldsymbol{R}$

$$\lim_{t\to\infty} \int_{t_0}^{t} \exp\left[-\int_{s}^{t} p_1(u)du\right]\left[\int_{t_0}^{s} H(u \pm k)du - M\right]ds = \pm\infty \tag{62}$$

(iii) the equation

$$u'' + p_1(t)u' + p_2(t)u = 0 \tag{63}$$

is disconjugate on $\boldsymbol{R}_+$ (for the definition of 'disconjugate', see [5]).
It is not difficult to verify that in order that the condition (C) in this paper be satisfied, we need only to assume that Eq. (63) is disconjugate. This shows that the conditions (i) and (ii) in ([5], theorem 4.1, theorem 4.2 and theorem 4.4) can be removed.

From the above remarks we see that this paper not only extends the related comparison theorems in [1]–[5] to more general equations, but also improves even for the special cases of this paper, the results in [1]–[5] to a large extent. In addition, the method in this paper is more normal, more general and more universal than that used in [1]–[5].

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