International Journal of Biomathematics
Vol. 7, No. 3 (2014) 1450030 (17 pages)

DOI: 10.1142/S1793524514500302

On a reaction–diffusion model for sterile insect release method on a bounded domain

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Received 16 July 2012
Accepted 14 March 2014
Published

We consider a system of partial differential equations that describes the interaction of the sterile and fertile species undergoing the Sterile Insect Release Method (SIRM). Unlike in the previous work [M. A. Lewis and P. Van Den Driessche, Waves of extinction from sterile insect release, *Math. Biosci.* **5** (1992) 221–247] where the habitat is assumed to be the one-dimensional whole space $\mathbb{R}$, we consider this system in a bounded one-dimensional domain (interval). Our goal is to derive sufficient conditions for success of the SIRM. We show the existence of the fertile-free steady state and prove its stability. Using the releasing rate as the parameter, and by a saddle-node bifurcation analysis, we obtain conditions for existence of two co-persistence steady states, one stable and the other unstable. Biological implications of our mathematical results are that (i) when the fertile population is at low level, the SIRM, even with small releasing rate, can successfully eradicate the fertile insects; (ii) when the fertile population is at a higher level, the SIRM can succeed as long as the strength of the sterile releasing is large enough, while the method may also fail if the releasing is not sufficient.

Keywords: Sterile insect release method; diffusion; saddle-node bifurcation; upper-lower solution method.

Mathematics Subject Classification 2010: 35B35, 35B40, 35B41, 35B51

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1. Introduction

The Sterile Insect Release Method (SIRM) is biological control method, which was first proposed by Knipling [17]. The idea is to release the insects of the same type that are sterilized and let them compete with the wild fertile insects for mating, so that the productive capacity of the target species is reduced by the competition. The method would be effective if the releasing can result in a population crash, controlling and even eradicating the insects. Due to its non-polluting nature and its effectiveness in certain situations, SIRM has become a well respected and very useful technique for pest control. This method was first implemented against screwworm fly (*Callitroga hominivorax*) in Curacao [5, 18, 21]. After that many species were successfully controlled via SIRM, such as the melon fly [13, 14], the codling moth [30], and the bollworm [11].

To guide the implementation of the SIRM or evaluate the effectiveness of the SIRM, mathematical models are useful. Barclay considered several SIRM models with effects of various factors. For example, he considered a model on SIRM with inter-specific competition in [1], and another model on SIRM with periodic release of parasitoids [2]. Barclay and Mackauer considered an SIRM model with the effect of predation or parasitism [4]. For the effect of predation on SIRM, one can also see [12]. The effect of the immigration of already inseminated females insect was considered for SIRM in [9, 29]. Readers can refer to other papers (e.g. [3, 6, 24, 28, 31]) for more information on certain aspects of SIRM. When the SIRM is applied in a large field, the diffusion of the species can not be ignored. Realizing the significance of spatial factor in pest control, [19, 23, 27] proposed partial differential equation (PDE) models with the diffusion terms accounting for mobility of the insects. Such models have revealed new phenomena that cannot be observed in ordinary differential equation models.

In [19], the authors considered an SIRM model given by the following system of reaction diffusion equations

$$\begin{cases} u_t = d_1\Delta u + u\left(\dfrac{a_1 u}{u+n} - a_2\right) - 2\delta u(u+n), \\ n_t = d_2\Delta n + r - a_2 n - 2\delta n(u+n), \end{cases} \quad t \geq 0,\ x \in \mathbb{R}. \tag{1.1}$$

Here $u(t,x)$ and $n(t,x)$ denote the densities of fertile and sterile females respectively, a_1 is the birth rate of the fertile insects, a_2 is the density-independent death rate which is smaller than a_1, and biologically realistic values for a_2/a_1 lie in the range $[0, 0.3]$. δ is the density-dependent death rate (given by $2\delta = (a_1 - a_2)/\text{carrying capacity}$). The constants d_1 and d_2 are the diffusion coefficients of fertile insects and sterile insects respectively, and r is the constant release rate of sterile insects. For details on the biological assumptions under which this model was proposed, see [19, 27]. By exploring existence of bistable traveling wave front solutions to (1.1), theoretically and numerically, some results were obtained on how the spatial diffusion together with other parameters affect the consequence of SIRM.

In the real world, a habit for a species of insects is bounded. This suggests that one should consider the PDE system in (1.1) in a bounded domain. In the one-dimensional case, a typical bounded domain is an interval. In this paper, we will consider the following initial-boundary value problem

$$\begin{cases} u_t = d_1 u_{xx} + u\left(\dfrac{a_1 u}{u+n} - a_2\right) - 2\delta u(u+n), & x \in \Omega, \\ n_t = d_2 n_{xx} + r - a_2 n - 2\delta n(u+n), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \dfrac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = 0, & t > 0, \\ u(x,0) = u_0(x) > 0, \quad n(x,0) = n_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where the meanings of the variables and parameters remain the same as in (1.1). For convenience yet without loss of generality, we choose the interval $[0, \ell]$ where $\ell > 0$, and consider a closed habitat reflected by the homogeneous Neumann boundary condition where $\frac{\partial}{\partial \nu}$ is the derivative along the outward normal direction. We are concerned with the dynamics of the model (1.2), in particular, under what conditions the population of the fertile species approaches zero, accounting for the success of the SIRM; and under what conditions, there will be a stable co-persistence steady state corresponding to the possibility of failure of the SIRM.

For convenience of analysis, we non-dimensionalize (1.2) by the following change of variables and parameters:

$$\begin{aligned} &x^* = x\sqrt{\frac{a_1}{d_1}}, \quad t^* = t a_1, \quad u^* = u\frac{\delta}{a_1}, \quad n^* = n\frac{\delta}{a_1}, \\ &A = \frac{a_2}{a_1}, \quad d = \frac{d_2}{d_1}, \quad R = \frac{\delta}{a_1^2} r, \quad L = \ell\sqrt{\frac{a_1}{d_1}}. \end{aligned} \tag{1.3}$$

For simplicity in later calculations, we let $\tilde{x} = \frac{\pi}{L} x^*$ and $C_1 = (\frac{\pi}{L})^2$, $C_2 = d(\frac{\pi}{L})^2$, so that the spatial domain is transformed into $\Omega^* = (0, \pi)$. Dropping asterisks and tildes for notational simplicity, (1.2) is then transformed into

$$\begin{cases} u_t = C_1 u_{xx} + u\left(\dfrac{u}{u+n} - A - 2(u+n)\right), & x \in \Omega, \\ n_t = C_2 n_{xx} + R - An - 2n(u+n), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \dfrac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = 0, & t > 0, \\ u(x,0) = u_0(x) > 0, \quad n(x,0) = n_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega = (0, \pi)$. We will point out later that system (1.4) is a competitive system. To study this kind of system, we can apply the theory and method introduced in [33]. For the work describing biological process with competitive system, one can see, for example, [7, 20, 22, 34].

The rest of the paper is organized as follows. We first address the well-posedness of the model (1.4) in Sec. 2. In Sec. 3, we show the existence of the fertile-free steady state for (1.4) and investigate its stability. In Sec. 4, we derive conditions for existence of co-persistence steady state for (1.4) and the global stability of the unique fertile-free steady state. We conclude the paper by Sec. 5 where we summarize our results and discuss their biological implications, as well as possible future works on the SIRM.

2. Well-Posedness of Model

In this section, we show that the model (1.4) is well-posed in the sense that for any pair of positive initial functions $(u_0(x), n_0(x))$, (1.4) has a unique solution which remains positive and bounded, and hence exists globally. To this end, we need some preparation.

For notational convenience, we denote by $f(u,n)$ and $g(u,n)$ the two nonlinear functions on the right-hand side of (1.4), that is,

$$f(u,n) \triangleq u\left(\frac{u}{u+n} - A - 2(u+n)\right),$$

$$g(u,n) \triangleq R - An - 2n(u+n).$$

It is easy to see that $g(u,n)$ is decreasing with respect to both u and n, and $g(u,0) > 0$, for all $u \in \mathbb{R}^+$. The form of $f(u,n)$ with respect to u is shown in Fig. 1, and $f(u,n) = 0$ has two positive solutions u_-, u_+ if and only if $n < n_c \triangleq (1-A)^2/8$ (see Fig. 1 for an illustration).

In order to consider classic solutions of (1.4), we introduce the space $X = (C^2(\Omega, \mathbb{R}^2) \cap C^1(\bar{\Omega}, \mathbb{R}^2))$. According to the biological meanings of the variables u and n, we only need to consider the following subset X^+ in X:

$$X^+ \triangleq \{(u,n) \in X \mid u \geq 0,\ n \geq 0,\ x \in \bar{\Omega}\}.$$

The following theorem confirms the well-posedness of (1.4), including existence, uniqueness, positivity and boundedness of a solution to (1.4).

Theorem 2.1. *For each* $(u_0, n_0) \in X^+$, *there exists a unique solution* $(u(t,x,u_0), n(t,x,n_0))$ *of system* (1.4) *and this solution remains in* X^+. *This solution is bounded, and hence, exists globally. Moreover, if* $u_0 \not\equiv 0$ *and* $n_0 \not\equiv 0$, *then* $u(t,x,u_0) > 0$ *and* $n(t,x,n_0) > 0$ *for all* $t > 0$ *and* $x \in \bar{\Omega}$.

Proof. Let $(u,n) \in \mathbb{R}^2_+$. It is easy to see that when $u = 0$, then $f(0,n) = 0 \geq 0$; and when $n = 0$, then $g(u,0) = R > 0$. By [33, Theorem 7.3.1 and Corollary 7.3.2], for each $(u_0, n_0) \in X^+$, there is a unique solution $(u(t,x,u_0), n(t,x,n_0))$ of system (1.4) and this solution remains in X^+ on a maximal time interval $[0, \sigma)$.

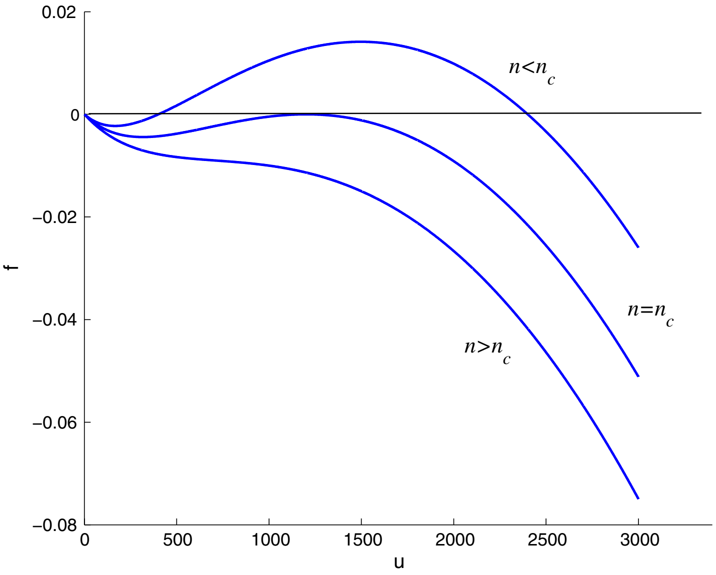

Fig. 1. Curves of f as a function of u when $A = 0.2$. Thus $n_c = 0.08$. Curves are shown for $n = 0.06$ (upper), $n = 0.08$, and $n = 0.1$ (lower).

In order to show the global existence of solutions of (1.4) (i.e. $\sigma = \infty$), we only need to show that solutions are bounded. Noticing that

$$f(u, n) \leq u(1 - A - 2u) = f(u, 0), \quad \text{for } (u, n) \in X^+,$$

we may consider the following upper comparison equation for the variable $u(t, x)$:

$$\begin{cases} u_t = C_1 u_{xx} + f(u, 0), & x \in \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{2.1}$$

Since every positive solution of (2.1) converges to the positive equilibrium $b_1 := (1 - A)/2$, by comparison theorem, we conclude that the u component of solution of (1.4) in X^+ remains bounded from above. Similarly,

$$g(u, n) \leq R - An - 2n^2 = g(0, n), \quad \text{for } (u, n) \in X^+,$$

suggesting that we consider the following upper comparison equation for the n component:

$$\begin{cases} n_t = C_2 u_{xx} + g(0, n), & x \in \Omega, \\ \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = 0, & t > 0, \\ n(x, 0) = n_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{2.2}$$

Again, by standard result on scalar reaction–diffusion equation with the Neumann boundary condition, we know that every positive solution of (2.2) approaches a positive constant steady state $b_2 := (\sqrt{A^2+8R}-A)/4$. By the comparison theorem, we conclude that the n component of any solution of (1.4) in X^+ remains bounded. Therefore, any solution of (1.4) in X^+ is bounded and hence exists globally.

The last part of the theorem on the strict positivity of solutions is a consequence of the Maximal Principle (see, e.g. [33, Corollary 7.2.3]). The proof is completed. □

Remark 2.2. By the proof of the above theorem and the [33, Corollary 7.3.3], we actually know that the set

$$X_b^+ = \{\phi \in X^+ : \phi(x) \in [0, b_1] \times [0, b_2]\}$$

is not only positively invariant but also attractive for (1.4).

3. Boundary Steady State: Eradication of Fertile Insects

Steady states play an important role in the dynamical properties of the system. Note that the results in [15, 16] imply that a cooperative system in a convex domain under the homogeneous Neumann boundary condition has no stable non-constant steady state. Thus, we only need to concentrate on constant steady states and their stability. To this end, we consider the points (u, n) such that $f(u, n) = g(u, n) = 0$.

For $f(u, n) = 0$, we have either $u = 0$, or

$$u - A(u+n) - 2(u+n)^2 = 0. \tag{3.1}$$

Equation (3.1) has two positive real roots for u if and only if $n < n_c$, given by

$$u_{\pm}(n) = \frac{1}{4}\left[(1-A) \pm \sqrt{(1-A)^2 - 8n}\right] - n > 0. \tag{3.2}$$

Or, equivalently, solving (3.1) for n in terms of u yields

$$n = n_f(u) \triangleq \frac{1}{4}\left(\sqrt{A^2 - 8u} - A - 4u\right). \tag{3.3}$$

For $g(u, n) = 0$, we have

$$u = u_g(n) \triangleq \frac{R - An}{2n} - n. \tag{3.4}$$

Combining $u = 0$ and (3.4), we have $n = b_2 = \frac{1}{4}(\sqrt{A^2+8R} - A)$. So we obtain a constant steady state $E_0 = (0, b_2)$, which will be called a fertile-free steady state at which, the fertile species becomes extinct.

Theorem 3.1. *The fertile-free steady state E_0 is always locally asymptotically stable.*

Proof. The stability of E_0 is determined by the following eigenvalue problem

$$\begin{cases} -C_1\phi_{xx} - f_u\phi - f_n\psi = \lambda\phi, \\ -C_2\psi_{xx} - g_u\phi - g_n\psi = \lambda\psi, \\ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = \left.\dfrac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0, \end{cases} \tag{3.5}$$

where all the partial derivatives are evaluated at $E_0 = (0, b_2)$, that is,

$$\begin{aligned} f_u &= -A - 2b_2, & f_n &= 0, \\ g_u &= -2b_2, & g_n &= -A. \end{aligned}$$

E_0 is locally asymptotically stable if and only if all the eigenvalues of (3.5) have positive real parts. Since the first equation of (3.5) is decoupled from the second one, the set of eigenvalues of (3.5) is a subset of the eigenvalues of the problem

$$\begin{cases} -C_1\phi_{xx} - f_u\phi = \lambda\phi, \\ \left.\dfrac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.6}$$

Because $-f_u = A + 2b_2 > 0$ on $\bar{\Omega}$, problem (3.6) has a simple principal eigenvalue λ_1 which is real and positive (see [10, Theorem 2.4]). Thus, all eigenvalues of (3.5) have positive real parts, confirming that E_0 is locally asymptotically stable. The proof is completed. ☐

The biological implication of this theorem is that applying the SIRM when the fertile species is at a lower density can always successfully eradicate the fertile species. However, when the fertile population is at a higher level, the success of the SIRM depends on the strength of the release rate R, and insufficient release may lead to co-existence of both fertile and sterile species, implying failure of the SIRM. We will explore this in the next section.

4. Co-Persistence Steady State: Failure of SIRM

We now consider possibility of co-persistence steady states. Substituting (3.4) into (3.3), we obtain

$$\begin{aligned} 0 = H(n) &\triangleq n_f(u_g(n)) - n \\ &= \frac{1}{4}\left(\sqrt{A^2 + \frac{4(R - An)}{n} - 8n} - A - \frac{2(R - An)}{n} + n\right) - n. \end{aligned} \tag{4.1}$$

This equation can be transformed to cubit equation, which has exactly one negative real root, with the other two being either real positive or a conjugate pair of complex numbers. The existence of two positive real roots requires that $R < \tilde{R}$, where $\tilde{R}$ is determined by solving the following two tangential equations:

$$H(n) = H'(n) = 0 \tag{4.2}$$

for some real $n > 0$.

Eliminating n from the two equations in (4.2), we can obtain

$$108R^2 + \tilde{A}R - A^2(1-A)^2 = 0, \tag{4.3}$$

where $\tilde{A} = 4(1+A)[A - 2(1-A)^2]$. This quadratic equation has a positive root (denoting it by $\tilde{R}$) which is given by

$$\tilde{R} = \frac{1}{216}\left(\sqrt{\tilde{A}^2 + 432A^2(1-A)^2} - \tilde{A}\right). \tag{4.4}$$

Corresponding to $R = \tilde{R}$, we can obtain the n value of the positive solution of (4.2):

$$\tilde{n} = \frac{1}{6}\left(\sqrt{A^2 + 6\tilde{R}(1+A)} - A\right). \tag{4.5}$$

Based on the above, we conclude that when $R > \tilde{R}$, (4.1) has no positive root, which implies that there is no positive constant steady state of (1.4). When $R = \tilde{R}$, there is exactly one positive constant steady state, denoted by $\tilde{E} = (u_+(\tilde{n}), \tilde{n})$. where $u_+(\tilde{n})$ is defined by (3.2). When $R < \tilde{R}$, (4.1) has exactly two positive roots denoted by n_- and n_+ satisfying $n_- < n_+$. See Fig. 2 for a demonstration of the above summary.

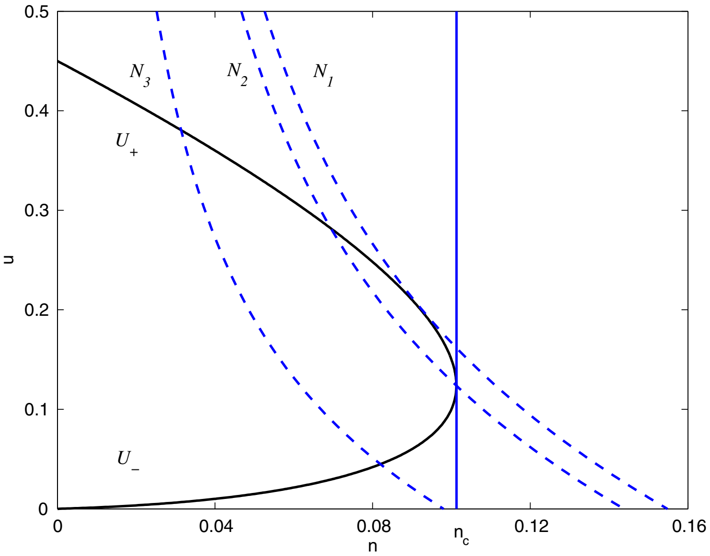

Fig. 2. The nullclines $f(u,n) = 0$ and $g(u,n) = 0$ on $n - u$ plane. Here $A = 0.1$, thus $n_c \approx 0.101$, $\tilde{R} = 0.0634$ and $R_c = 0,0557$. U_+ and U_- are the curve described by (3.2); N_1, N_2, and N_3 are the curves described by (3.4) for $R = \tilde{R}$, $R = R_c$ and $R = 0.029 < R_c$, respectively.

It is easy to see that when $R < \tilde{R}$, the n components of the two positive steady states, n_+ and n_-, may locate either on the same branch (U_+) or on different branches (U_+ and U_-). Obviously, there is another critical value $R_c \in (0, \tilde{R})$, given by $R_c = (1+A)(1-A)^2/16$, that distinguishes these two cases (see Fig. 2). That is, for $R \in (R_c, \tilde{R})$, the two positive constant steady states are given by

$$E_1 = (u_+(n_-), n_-), \quad E_2 = (u_+(n_+), n_+),$$

while for $R \in (0, R_c)$, they are given by

$$E_1 = (u_+(n_-), n_-), \quad E_2 = (u_-(n_+), n_+).$$

Basing on the discussion above, we have following theorem.

Theorem 4.1. *For given parameters A and d, the following results for* (1.4) *hold.*

(i) *When $R > \tilde{R}$, there is no positive constant steady state for* (1.4). *Moreover, the unique constant steady state E_0 is globally asymptotically stable.*

(ii) *When $R = \tilde{R}$, there is exactly one positive constant steady state $\tilde{E} = (u_+(\tilde{n}), \tilde{n})$. Moreover, system* (1.4) *undergoes a saddle-node bifurcation at point $\tilde{E}$.*

(iii) *When $R_c < R < \tilde{R}$, there are exactly two positive constant steady states for* (1.4), *given by $E_1 = (u_+(n_-), n_-)$ and $E_2 = (u_+(n_+), n_+)$. Moreover, E_1 is locally asymptotically stable and E_2 is unstable.*

(iv) *When $0 < R \leq R_c$, there are exactly two positive constant steady states for* (1.4), *given by $E_1 = (u_+(n_-), n_-)$ and $E_2 = (u_-(n_+), n_+)$. Moreover, E_1 is locally asymptotically stable and E_2 is unstable.*

Before giving the proof of Theorem 4.1, for readers' convenience, we present the well known Saddle-Node Bifurcation Theorem by Crandall and Rabinowitz in [8].

Theorem 4.2. *Let X, Y be Banach spaces and $\mathscr{F} : \mathbb{R} \times X \to Y$. Assume that there is $(\lambda_0, U_0) \in \mathbb{R} \times X$ such that $\mathscr{F}(\lambda_0, U_0) = 0$ and $\mathscr{F}$ is a continuously differentiable in an open neighborhood of (λ_0, U_0). Suppose the following conditions hold*:

(H1) $\dim \mathscr{N}(\mathscr{F}_U(\lambda_0, U_0)) = \operatorname{codim} \mathscr{R}(\mathscr{F}_U(\lambda_0, U_0)) = 1$ *and* $\mathscr{N}(\mathscr{F}_U(\lambda_0, U_0)) = \operatorname{span}\{\Phi_0\}$;

(H2) $\mathscr{F}_\lambda(\lambda_0, U_0) \notin \mathscr{R}(\mathscr{F}_U(\lambda_0, U_0))$.

Let Z be the complement of $\operatorname{span}\{\Phi_0\}$ *in X. Then the solutions of $\mathscr{F}(\lambda, U) = 0$ near (λ_0, U_0) form a curve $(\lambda(s), U(s)) = (\lambda_0 + \tau(s), U_0 + s\Phi_0 + sz(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ satisfying $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. Furthermore, if $\mathscr{F}$ is k-times continuously differentiable, so are $\tau(s)$ and $z(s)$.*

For work on studies and applications of saddle-node bifurcation, one can see, for example, [25]. For some extended results for Theorem 4.2, one can see [32].

Now we give the proof of Theorem 4.1.

Proof of Theorem 4.1. The assertions on existence/non-existence of positive constant steady states have already been discussed above. Now we confirm the conclusions on stability/instability of the steady states, as well as on the saddle-note bifurcation.

We know that the operator $\phi \mapsto -\phi''$ on $(0, \pi)$ with boundary condition $\phi'(0) = \phi'(\pi) = 0$ has eigenvalues

$$\mu_0 = 0, \quad \mu_k = k^2, \quad k = 1, 2, 3, \ldots,$$

with the associated normalized eigenfunctions

$$\phi_0(x) = \sqrt{\frac{1}{\pi}}, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}}\cos(kx), \quad k = 1, 2, 3, \ldots,$$

respectively. The linearized system of (1.4) at one steady state has the form:

$$\begin{pmatrix} u_t \\ n_t \end{pmatrix} = L\begin{pmatrix} u \\ n \end{pmatrix} \triangleq C\begin{pmatrix} u_{xx} \\ n_{xx} \end{pmatrix} + J\begin{pmatrix} u \\ n \end{pmatrix}, \tag{4.6}$$

where

$$C \triangleq \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

and J is the Jacobian matrix defined by

$$J \triangleq \begin{pmatrix} f_u & f_n \\ g_u & g_n \end{pmatrix}.$$

Here all derivatives will be evaluated at the steady state under consideration. Obviously, L is a linear operator with domain $D_L = X_{\mathbb{C}} \triangleq \tilde{X} \oplus i\tilde{X} = \{\mathbf{v}_1 + i\mathbf{v}_2 \,|\, \mathbf{v}_1, \mathbf{v}_2 \in \tilde{X}\}$, where $\tilde{X} \triangleq \{(u, n) \in X \,|\, u'(0) = u'(\pi) = n'(0) = n'(\pi) = 0\}$ is a real-valued Sobolev space.

Consider the following characteristic equation of the operator L:

$$L\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mu\begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{4.7}$$

The steady state is locally asymptotically stable if and only if all the eigenvalues of (4.7) have negative real parts. Let $(\phi(x), \psi(x))^\top$ be an eigenfunction of L corresponding to the eigenvalue μ, which can be expressed as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos(kx),$$

where a_k, $b_k \in \mathbb{C}$ are coefficients. Plugging the above into (4.7) gives

$$-C\sum_{k=0}^{\infty} k^2 \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx + J\sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx = \mu\sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos kx,$$

which leads to

$$(J - k^2C)\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \mu \begin{pmatrix} a_k \\ b_k \end{pmatrix}, \quad k = 0, 1, 2, \ldots.$$

Set

$$J_k \triangleq J - k^2D = \begin{pmatrix} f_u - k^2C_1 & f_n \\ g_u & g_n - k^2C_2 \end{pmatrix}, \quad k = 0, 1, 2, \ldots.$$

It is clear that the eigenvalues of L are given by the eigenvalues of J_k for $k = 0, 1, 2, \ldots$. The characteristic equation of J_k is

$$(Q)_k : \quad \mu^2 - T_k\mu + D_k = 0, \quad k = 0, 1, 2, \ldots,$$

where

$$T_k \triangleq \operatorname{tr} J_k = f_u + g_n - k^2(C_1 + C_2),$$

$$D_k \triangleq \det J_k = C_1C_2k^4 - (C_2f_u + C_1g_n)k^2 + f_ug_n - f_ng_u.$$

Then all the eigenvalues of (4.7) are given by the roots of $(Q)_k,\ k = 0, 1, 2, \ldots$. We can determine the sign of the real parts of the eigenvalues of $(Q)_k$ by analyzing the corresponding T_k and D_k.

At the point $\tilde{E}$, $f_u < 0$, $f_n < 0$, $g_u < 0$ and $g_n < 0$, implying that D_k is increasing in k for $k \geq 0$ and hence, $D_k > D_0$ for all $k = 1, 2, \ldots$. On the other hand, from (4.1) and (4.2), we obtain

$$\begin{aligned} 0 = H'(\tilde{n}) &= \frac{dn_f(u)}{du} \cdot \frac{du_g(\tilde{n})}{dn} - 1 \\ &= \left(-\frac{f_u(u_+(\tilde{n}), \tilde{n})}{f_n(u_+(\tilde{n}), \tilde{n})}\right) \cdot \left(-\frac{g_n(u_+(\tilde{n}), \tilde{n})}{g_u(u_+(\tilde{n}), \tilde{n})}\right) - 1. \end{aligned}$$

Hence $D_0 = f_ug_n - f_ng_u = 0$ at $\tilde{E}$. Since $T_k < T_0 = f_u + g_n < 0,\ k = 0, 1, 2, \ldots$, so $(Q)_0$ has a negative real root and a zero root; for $k = 1, 2, \ldots$ all roots of $(Q)_k$ have negative real parts.

To prove that system (1.4) undergoes saddle-node bifurcation at $\tilde{E}$ when $R = \tilde{R}$, we only need to verify conditions (H1) and (H2) in Theorem 4.2. For similar process, one can see [25, Proof of Theorem 2.2]. Corresponding to the notations in Theorem 4.2, for our model, we chose the mapping $\mathscr{F} : \mathbb{R}_+ \times X_{\mathbb{C}} \to C^\alpha(\Omega)$ as

$$\mathscr{F}(R, (u, n)) = \begin{pmatrix} C_1u_{xx} + f(u, n) \\ C_2n_{xx} + g(u, n) \end{pmatrix}.$$

It is clear that $\mathscr{F}(\tilde{R}, \tilde{E}) = 0$ and the derivative $\mathscr{F}_R(\tilde{R}, \tilde{E}) = (0, 1)$. We can also easily calculate the derivative of $\mathscr{F}$ with respect to E at $(\tilde{R}, \tilde{E})$ as

$$\mathscr{F}_E(\tilde{R}, \tilde{E})[(\phi, \psi)] = \begin{pmatrix} C_1\phi_{xx} + f_u\phi + f_n\psi \\ C_2\psi_{xx} + g_u\phi + g_n\psi \end{pmatrix}, \quad (\phi, \psi) \in X_{\mathbb{C}}.$$

To find the null space of $\mathscr{F}_E(\tilde{R}, \tilde{E})$, we need to solve $\mathscr{F}_E(\tilde{R}, \tilde{E})[(\phi, \psi)] = 0$ for non-trivial solutions. We can regard this question as an eigenvalue problem with eigenvalue $\lambda = 0$. Let $\phi(x) = \sum_{k=0}^{\infty} a_k \cos kx$, $\psi(x) = \sum_{k=0}^{\infty} b_k \cos kx$, a_k, $b_k \in \mathbb{C}$. Substituting $(\phi(x),\psi(x))$ into $\mathscr{F}_E(\tilde{R},\tilde{E})[(\phi,\psi)] = 0$, we can obtain

$$\begin{cases} a_k = \dfrac{f_n}{C_1k^2 - f_u} b_k, \\ \left[C_1C_2k^2 - (C_2f_u + C_1g_n)\right] k^2 b_k = 0, \end{cases} \qquad k = 0, 1, 2, \ldots.$$

By $f_n < 0$ and $f_u < 0$ (hence $C_1k^2 - f_u > 0$ and $C_2f_u + C_1g_n < 0$), we conclude that $b_k = 0$, for all $k = 1, 2, \ldots$ and $a_0 = -\frac{f_n}{f_u}b_0$ where $b_0 \neq 0$ and hence, $\psi(x) \equiv b_0$ and $\phi(x) \equiv a_0 = -\frac{f_n}{f_u}b_0$. This implies that $\dim \mathscr{N}(\mathscr{F}_E(\tilde{R},\tilde{E})) = 1$ and $\mathscr{N}(\mathscr{F}_E(\tilde{R},\tilde{E})) = \operatorname{span}\{(-\frac{f_n}{f_u}, 1)\}$.

For each $(w,z) \in \mathscr{R}(\mathscr{F}_E(\tilde{R},\tilde{E}))$, there exists $(\phi,\psi) \in X_{\mathbb{C}}$ such that

$$C_1\phi_{xx} + f_u\phi + f_n\psi = w, \tag{4.8}$$

$$C_1\phi_{xx} + g_u\phi + g_n\psi = z. \tag{4.9}$$

Multiplying (4.8) by g_n and (4.9) by $-f_n$, and then subtracting the two resulting equations lead to

$$C_1g_n\phi_{xx} - C_2f_n\psi_{xx} = (g_nw - f_nz) = (w,z)(g_n, -f_n)^{\top}.$$

Here we have made use of the relation $f_ug_n - g_uf_n = D_0 = 0$. Integrating the above equation and using the Neumann boundary condition, we obtain

$$0 = \int_{\Omega} (g_nC_1\phi_{xx} - f_nC_2\psi_{xx})dx = \int_{\Omega} (g_nw - f_nz)dx = \int_{\Omega} (w,z)(g_n,-f_n)^{\top}dx. \tag{4.10}$$

On the other hand, by the Fredholm's theorem, if $\int_{\Omega}(w,z)(g_n,-f_n)^{\top}dx = 0$ holds for $(w,z) \in C^{\alpha}(\Omega)$, then there exists a $(\phi,\psi) \in X_{\mathbb{C}}$ satisfying (4.8)–(4.9). Thus, $\mathscr{R}(\mathscr{F}_E(\tilde{R},\tilde{E})) = \{(w,z) \in C^{\alpha}(\Omega) : \int_{\Omega}(w,z)(g_n,-f_n)^{\top}dx = 0\}$, which shows $\operatorname{codim}\mathscr{R}(\mathscr{F}_E(\tilde{R},\tilde{E})) = 1$. Thus $\mathscr{F}$ satisfies (H1).

Now we show that $\mathscr{F}_R = (0,1) \notin \mathscr{R}(\mathscr{F}_E(\tilde{R},\tilde{E}))$. If this is not true, then there exists a $(\phi,\psi) \in X_{\mathbb{C}}$ such that

$$C_1\phi_{xx} + f_u\phi + f_n\psi = 0, \tag{4.11}$$

$$C_1\psi_{xx} + g_u\phi + g_n\psi = 1. \tag{4.12}$$

From (4.11), we have $\phi = -(C_1\phi_{xx} + f_n\psi)/f_u$. Substituting this into (4.12), integrating the resulting equation and making use of the fact $g_n - g_uf_n/f_u = 0$ at $(\tilde{R},\tilde{E})$, we are led to

$$0 = \int_{\Omega} \left(C_2\psi_{xx} - \frac{g_u}{f_u}C_1\phi_{xx}\right) dx = \int_{\Omega} 1dx > 0,$$

a contradiction. Thus, $\mathscr{F}$ satisfies (H2). By Theorem 4.2, system (1.4) undergoes a saddle-node bifurcation at point $\tilde{E}$ when $R = \tilde{R}$.

Now we consider the stability of E_1 and E_2 when $R \in (R_c, \tilde{R})$. We first show that D_0 defined in $(Q)_0$ is decreasing with respect to n on the curve $n = n_f(u)$. In fact, for a steady state on the curve $n = n_f(u)$

$$\begin{aligned} D_0 &= f_u g_n - f_n g_u \\ &= u\left(\frac{n}{(u+n)^2} - 2\right) \cdot (-2u - 4n) - u\left(\frac{u}{(u+n)^2} + 2\right) \cdot 2n \\ &= -4u\left(\frac{n}{u+n} - (u+n)\right) \end{aligned}$$

and

$$\frac{\partial D_0}{\partial n} = -4u\left(\frac{u - (u+n)^2}{(u+n)^2}\right). \tag{4.13}$$

On the curve $n = n_f(u)$, (u, n) satisfies $u - A(u+n) - 2(u+n)^2 = 0$, so $u > (u+n)^2$, which implies that $\frac{\partial D_0}{\partial n}|_{n=n_f(u)} < 0$. Note that when $R \in (R_c, \tilde{R})$, $E_1 = (u_+(n_-), n_-)$ and $E_2 = (u_+(n_+), n_+)$ with $n_- < \tilde{n} < n_+$. This together with the fact that $D_0|_{\tilde{E}} = 0$ implies that $D_0|_{E_1} > 0$ and $D_0|_{E_2} < 0$. Therefore, E_1 is locally asymptotically stable since $T_k|_{E_1} < T_0|_{E_1} = f_u + g_n < 0$ and $D_k|_{E_1} > D_0|_{E_1} > 0$, and E_2 is unstable since $D_0|_{E_2} < 0$.

When $R \in (0, R_c)$, E_1 is still given by $E_1 = (u_+(n_-), n_-)$ with $n_- < \tilde{n}$. So E_1 is still locally asymptotically stable in this case. For $E_2 = (u_-(n_+), n_+)$, we also have $D_0|_{E_2} < 0$ since $f_u > 0$, $f_n < 0$, $g_u < 0$ and $g_n < 0$. This implies that there is at least one positive real eigenvalue of (4.7). Thus E_2 is unstable.

Next we prove the *global* stability of E_0 when $R > \tilde{R}$. Noticing that system (1.4) is a two-dimensional competitive system, it can be viewed as a monotone dynamical system with respect to the partial ordering $\leq_K$ induced by the second quadrant cone:

$$K = \{(u, n) \in X : u \leq 0,\ n \geq 0\},$$

that is,

$$(u_1, n_1) \leq_K (u_2, n_2) \Leftrightarrow u_1 \geq u_2 \quad \text{and} \quad n_1 \leq n_2.$$

Alternatively, by the change of variables $v = -u$, (1.4) is transformed into a cooperative system

$$\begin{cases} v_t = C_1 v_{xx} - v\left(\dfrac{v}{n-v} + A + 2(n-v)\right), & x \in \Omega, \\ n_t = C_2 n_{xx} + R - An - 2n(n-v), & x \in \Omega, \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\partial\Omega} = \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = 0, & t > 0, \\ v(x,0) = v_0(x) < 0, \quad n(x,0) = n_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{4.14}$$

with a unique constant steady state $E_0' = (0, b_2)$ when $R > \tilde{R}$. The invariant set X^+ for (1.4) is obviously transformed to the invariant set X' for (4.14) where

$$X' \triangleq \{(v, n) \in X : v \le 0, n \ge 0, \forall\, x \in \bar{\Omega}\},$$

with the natural partial order $\preceq$, that is $(v_1, n_1) \preceq (v_2,\ n_2) \Leftrightarrow v_1 \le v_2,\ n_1 \le n_2$. Associate to (4.14) is the following cooperative system of ordinary differential equations

$$\begin{cases} \dfrac{dv}{dt} = -v\left(\dfrac{v}{n-v} - A + 2(n-v)\right), \\ \dfrac{dn}{dt} = R - An - 2n(n-v), \end{cases} \tag{4.15}$$

By similar arguments to that in the proof of Theorem 2.1, we know that the solutions of (4.15) are eventually bounded, and the rectangular domain

$$\Lambda' \triangleq \left\{(v, n) \in \mathbb{R}^2 : \frac{1-A}{2} \le v \le 0, 0 \le n \le b_2\right\}$$

is an invariant set and attracts all the solutions starting in the second quadrant (see also Remark 2.2). When $R > \tilde{R}$, E_0' is the unique equilibrium for (4.15) in Λ'. System (4.15) has no periodic orbit since it is a planar and cooperative system. By Poincaré–Bendixson theorem, E_0' is globally stable for (4.15).

For any given $(v_0(x), n_0(x)) \in X'$, let $\Psi(t, v_0, n_0)$ is the solution of (4.14) with initial value $(u_0(x), n_0(x))$. Let

$$\bar{v}_0 = \sup_{x\in\bar{\Omega}} v_0(x), \quad \underline{v}_0 = \inf_{x\in\bar{\Omega}} v_0(x),$$
$$\bar{n}_0 = \sup_{x\in\bar{\Omega}} n_0(x), \quad \underline{n}_0 = \inf_{x\in\bar{\Omega}} n_0(x),$$

and denote by $\Phi(t, \bar{v}_0, \bar{n}_0)$, $\Phi(t, \underline{v}_0, \underline{n}_0)$ the solutions of (4.15) with initial values $(\bar{v}_0, \bar{n}_0)$ and $(\underline{v}_0, \underline{n}_0)$, respectively. Note that $\Phi(t, \bar{v}_0, \bar{n}_0)$ and $\Phi(t, \underline{v}_0, \underline{n}_0)$ also satisfy the PDEs with the Neumann boundary condition in (4.14). Then, by the comparison principle for cooperative parabolic systems, we then have

$$\Phi(t, \underline{v}_0, \underline{n}_0) \preceq \Psi(t, x, v_0, n_0) \preceq \Phi(t, \bar{v}_0, \bar{n}_0).$$

The convergence of $\Phi(t, \underline{v}_0, \underline{n}_0)$ and $\Phi(t, \bar{v}_0, \bar{n}_0)$ to E_0' implies that $\Psi(t, x, v_0, n_0) \to E_0'$, as $t \to +\infty$. That is, E_0' is globally asymptotically stable for (4.14) in X'. Translating this conclusion in terms of original system (1.4), we conclude that the fertile-free steady state E_0 is globally asymptotically stable for (1.4), completing the proof. ☐

5. Conclusion and Discussion

In this paper, we have considered a spatially diffusive SIRM model which release evenly a bounded habitat. The model is an alternation of the one considered in [19] in the sense that the unbounded domain (interval) is replaced by an bounded domain (interval). We have confirmed the well-posedness of the model and investigated the dynamics of the model by analyzing all possible steady states and their stability. The results are described in terms of the unique artificial parameter R, the release strength. This seems to be a natural choice because what people can control in implementing the SIRM is the amount of release, while all other parameters, such as birth rate, death rate, environment carrying capacity and diffusion rate are determined intrinsically by the environment and insect species itself. Our results show that when the fertile population is at low level, a small release rate R can eradicate the fertile population (local stability of E_0 in Theorem 3.1); however, when the fertile population has grown up to a high level, insufficient release rate may fail to eradicate the fertile species resulting to co-existence of both the fertile and sterile species (stability of E_1 in Theorem 4.1).

The model (1.4) adopts the mechanism of evenly releasing domain-wise used in [19]. Such a mechanism seems to be very hard and challenging, if not impossible, to implement in practice. A more realistic mechanism would be releasing on the boundary only. Replacing the evenly domain-wise releasing by a boundary releasing will lead to the following alternation of (1.4):

$$\begin{cases} u_t = C_1 u_{xx} + u\left(\dfrac{u}{u+n} - A - 2(u+n)\right), & x \in \Omega, \\ n_t = C_2 n_{xx} - An - 2n(u+n), & x \in \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial n}{\partial \nu}\right|_{\partial\Omega} = R, & t > 0, \\ u(x,0) = u_0(x) > 0, \quad n(x,0) = n_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{5.1}$$

In a forthcoming work, we will analyze this new model and investigate whether or not this new and easier releasing mechanism can still successfully eradicate the fertile insect species.

Acknowledgments

Part of this work was completed when the second author was visiting the University of Western Ontario, and he would like to thank the staff in the Department of Applied Mathematics for their help and thank the university for its excellent facilities and support during his stay.

The second author was supported by China Scholarship Council, partially supported by NNSF of China (No. 11031002), by the Heilongjiang Provincial Natural Science Foundation (No. A200806), and by the Program of Excellent Team and the Science Research Foundation in Harbin Institute of Technology.

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