Journal of Dynamics and Differential Equations, Vol. 13, No. 3, 2001

Traveling Wave Fronts of Reaction-Diffusion Systems with Delay

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Received February 16, 1999

This paper deals with the existence of traveling wave front solutions of reaction-diffusion systems with delay. A monotone iteration scheme is established for the corresponding wave system. If the reaction term satisfies the so-called quasi-monotonicity condition, it is shown that the iteration converges to a solution of the wave system, provided that the initial function for the iteration is chosen to be an upper solution and is from the profile set. For systems with certain non-quasimonotone reaction terms, a convergence result is also obtained by further restricting the initial functions of the iteration and using a non-standard ordering of the profile set. Applications are made to the delayed Fishery–KPP equation with a nonmonotone delayed reaction term and to the delayed system of the Belousov–Zhabotinskii reaction model.

KEY WORDS: Traveling wave fronts; reaction-diffusion systems with delay; monotone iteration; nonstandard ordering; quasimonotonicity; nonquasimonotonicity.
AMS Subject Classifications: 34K10, 35B20, 35K57.

1. INTRODUCTION

In this paper, we study the existence of traveling fronts of delayed reaction-diffusion systems using various results and methods in the theory of monotone dynamical systems.

For illustration, we first consider the following well-studied Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad x \in \mathbb{R} \tag{1.1}$$

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1040-7294/01/0700-0651$19.50/0

A traveling wave front is a solution with the special form $u(t, x) = \phi(x - ct)$ for some constant $c > 0$ and for some twice continuously differentiable function ϕ: $\mathbb{R} \to \mathbb{R}$ satisfying the asymptotic boundary conditions $\lim_{s \to -\infty} \phi(s) = 0$ and $\lim_{s \to \infty} \phi(s) = 1$. Clearly, with respect to the wave variable $s = x - xt$, the profile ϕ is given by the following second-order ordinary differential equation:

$$-c\dot{\phi} = \ddot{\phi} + \phi(1 - \phi) \tag{1.2}$$

It is well-known that for every $c \geqslant 2$, Eq. (1.2) has a solution with the required asymptotic boundary values. For motivations, let us reproduce this result using an existence result for heteroclinic orbits of monotone dynamical systems. Let

$$\begin{cases} x_1 = \phi \\ x_2 = \phi + \alpha\dot{\phi} \end{cases} \tag{1.3}$$

where $\alpha > 0$ is a constant to be determined. Then, we get

$$\begin{cases} \dot{x}_1 = -\dfrac{1}{\alpha} x_1 + \dfrac{1}{\alpha} x_2 := f_1(x_1, x_2) \\ \dot{x}_2 = \dot{\phi} + \alpha\ddot{\phi} = (1 - \alpha c)\, \dot{\phi} - \alpha\phi(1 - \alpha) \\ \quad = \dfrac{1 - \alpha c}{\alpha} (x_2 - x_1) - \alpha x_1(1 - x_1) := f_2(x_1, x_2) \end{cases} \tag{1.4}$$

Note that system (1.4) has two ordered equilibria, given by $(0, 0)^T$ and $(1, 1)^T$. Moreover,

$$\begin{cases} \dfrac{\partial f_1}{\partial x_2} = \dfrac{1}{\alpha} > 0 \\ \dfrac{\partial f_2}{\partial x_1} = \dfrac{\alpha c - 1}{\alpha} - \alpha + 2\alpha x_1 \geqslant c - \left(\alpha + \dfrac{1}{\alpha} \right) \end{cases}$$

provided that $0 \leqslant x_1 \leqslant 1$. Therefore, for any $c \geqslant 2$ we can always find $\alpha > 0$ so that $\partial f_2(x_1, x_2)/\partial x_1 \geqslant c - (\alpha + (1/\alpha)) \geqslant 0$ in the interval where $0 \leqslant x_1 \leqslant 1$ and $0 \leqslant x_2 \leqslant 1$. [Note that the minimal wave speed $c = 2$ corresponds exactly to the minimal value of the function $\alpha + (1/\alpha)$!] In other words, for each $c \geqslant 2$ we can choose $\alpha > 0$ such that system (1.4) possesses an ordered pair of equilibria and, simultaneously, generates a monotone flow in the ordered interval $0 \leqslant x_1 \leqslant 1$ and $0 \leqslant x_2 \leqslant 1$ (here and in what follows, the ordering is the usual componentwise ordering). Consequently, a well-known result [see, e.g., Smith (1995), p. 27] ensures the existence of a heteroclinic orbit connecting $(0, 0)^T$ and $(1, 1)^T$. This establishes the existence of a wave front for (1.1) with $c \geqslant 2$.

The crucial step in the above argument is the simple linear transformation (1.3), which serves two purposes simultaneously: (i) to transform the two equilibria $(0, 0)^T$ and $(1, 0)^T$ of (1.2) into an ordered pair $(0, 0)^T$ and $(1, 1)^T$ for system (1.4) and (ii) to transform Eq. (1.2) into a cooperative system (1.4) which generates a monotone flow. The idea can, of course, be extended to general systems of reaction-diffusion equations

$$\frac{\partial u(t, x)}{\partial t} = D \frac{\partial^2 u(t, x)}{\partial x^2} + f(u(t, x)) \tag{1.5}$$

where $x \in \mathbb{R}$, $u \in \mathbb{R}^n$, $D = diag(d_1, ..., d_n)$ with $d_i > 0$ for $1 \leqslant i \leqslant n$, $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1, $f(\mathbf{0}) = f(K) = \mathbf{0}$ for a vector $K \in \mathbb{R}^n$ with positive components, and f has no other zero in the ordered interval $\mathbf{0} \leqslant u \leqslant K$. Using basically the same argument as above for the simple Fisher equation (1.1), we can obtain the following general result.

Theorem A. *Assume that* $\partial f_i(u)/\partial u_j \geqslant 0$ *for* $\mathbf{0} \leqslant u \leqslant K$ *and* $1 \leqslant i \neq j \leqslant n$. *Let*

$$c^* = \inf_{b_i > 0,\, 1 \leqslant i \leqslant n} \ \sup_{\mathbf{0} \leqslant u \leqslant K} \left[\frac{d_i}{b_i} + b_i \frac{\partial f_i(u)}{\partial u_i} \right]$$

Then for each $c > c^*$, *system (1.5) has a wave front* $u(t, x) = \phi(x - ct)$ *such that* $\lim_{s \to \infty} \phi(s) = \mathbf{0}$ *and* $\lim_{s \to \infty} \phi(s) = K$.

It is natural to ask if the above results and methods can be extended to general reaction-diffusion systems with delay, a prototype of which takes the form

$$\frac{\partial u(t, x)}{\partial t} = D \frac{\partial^2 u(t, x)}{\partial x^2} + f(u(t, x), u(t - \tau, x)) \tag{1.6}$$

where $\tau \geqslant 0$ is a given constant and $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Unfortunately, if we look for a wave front $u(t, x) = \phi(x - ct)$ with $c > 0$, then we obtain the following second order functional differential equation of advanced type for the profile ϕ

$$-c\dot{\phi}(s) = D\ddot{\phi}(s) + f(\phi(s), \phi(s + c\tau)) \tag{1.7}$$

which does not generate a semiflow. Reversing the wave variable s [or, equivalently, looking for a wave front of the form $u(t, x) = \phi(x + ct)$ with $c > 0$] could lead to a functional differential equation for which the machinery was developed by Smith (1987) in order to apply the theory of monotone dynamical system, but it seems difficult, if not impossible, to construct a linear transformation of the form $Y = \phi$, $Z = \phi + B\dot{\phi}$ (for some $n \times n$ matrix B) so that the transformed system (1.7) generates a monotone semiflow and simultaneously possesses an ordered pair of equilibria corresponding to the steady-states of (1.6)! This difficulty, due to the presence of delay in the reaction term, prevents us from directly applying the known results in the powerful (and modern) theory of monotone dynamical systems to tackle the existence of traveling wave fronts for delayed reaction-diffusion systems.

It is the purpose of this paper to show that, despite the above difficulty, the classical monotone iteration technique coupled with the upper-lower solutions provides an effective tool in establishing the existence of wave fronts for delayed reaction-diffusion systems, at least for certain classes of reaction nonlinearities enjoying some monotonicity properties. In particular, we develop a monotone iteration scheme and establish the convergence of this scheme to a wave front if an ordered pair of upper-lower solutions exists and if the nonlinearity satisfies the so-called quasimonotonicity condition previously used in different context by, for example, Ahmod and Vatsala (1981), Kerscher and Nagel (1984), Kunish and Schappacher (1979), Ladas and Lakshmikanthan (1972), Lakshmikanthan and Leela (1981), Leela and Moauro (1978), Martin and Smith (1990, 1991), and Smith (1987, 1991). In the case where the nonlinearity does not satisfy the quasimonotonicity condition, we employ the nonstandard ordering of the profile set, previously introduced by Smith and Thieme (1990, 1991) in order for ordinary functional differential equations to generate strongly monotone semiflows. With further technical restrictions on the ordered pair of upper-lower solutions, we are able to establish the monotonicity and convergence of the iteration scheme and, thus, obtain the existence of traveling wave fronts.

We illustrate our general results with two examples. One is the delayed diffusive logistic equation

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x)[1 - u(t - \tau, x)] \tag{1.8}$$

and another example is the following Belousov–Zhabotinskii reaction model with discrete time lag,

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)[1 - u(t,x) - rv(t-\tau,x)] \\ \dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial^2 v(t,x)}{\partial x^2} - bu(t,x)\, v(t,x) \end{cases} \tag{1.9}$$

Applying our general results involving nonstandard ordering of the profile set, we can show that for $c > 2$ there exists $\tau^*(c) > 0$ such that if $0 \leqslant \tau \leqslant \tau^*(c)$, then (1.8) has a traveling wave front with the wave speed c. Our result strongly indicates that if a scalar reaction-diffusion system,

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + g(u(t,x), u(t,x)) \tag{1.10}$$

has a wave front, then so does the delayed analogue,

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + g(u(t,x), u(t-\tau,x))$$

provided the delay is small. Model (1.9) possesses the quasimonotonicity property after a change of variable $v \to 1 - v$. Applications of our general results to (1.9) not only reproduce but also improve the existence results obtained by Troy (1980), Ye and Wang (1987), Kanel (1990), and Kapel (1991) even in the case $\tau = 0$. For example, Troy (1980) has claimed that if $b \in (0, 1)$, then there exist $c^* \in (0, 2)$ and $r^* > 0$ such that (1.9) with $r = r^*$ has a wave front with the speed c^*. Our study shows that if $b \in (0, 1)$, then for every $c \in [2\sqrt{1 - r^*}, 2]$, where r^* is chosen so that $0 < b < 1 - r^*$, (1.9) has a wave front with the speed c. Moreover, our results show that the existence of wave fronts is independent of the size of the delay.

It should be mentioned that traveling wave solutions for reaction-diffusion equations without delay have been extensively studied in the literature. The recent book review by Gardner (1995) and the monographs by Fife (1979), Britton (1986), Murray (1989), and Volpert *et al.* (1994) provide a full discussion of the subject. On the other hand, time delay should be and has been incorporated into reaction terms in many realistic models in applications. Some progress has been made for the existence and qualitative theory of wave fronts of delayed reaction-diffusion systems. The recent monograph by Wu (1996) provides a brief account of the progress. But we should particularly mention the pioneering work of Schaaf (1987), where scalar reaction-diffusion equations with a discrete delay were studied, using the phase-plane technique, the maximum principle for parabolic functional differential equations, and the general theory for ordinary functional differential equations. Unfortunately, the work of Schaaf (1987) applies only to a scalar delayed reaction-diffusion equation where the nonlinearity either is of the Hodgkin–Huxley type or satisfies the quasimonotonicity condition.

The remaining part of this paper is organized as follows. Section 2 is devoted to some preliminary discussion. In Section 3, we develop a monotone iteration scheme and prove its convergence to a wave front if the nonlinear reaction term satisfies the so-called quasimonotonicity condition. In Section 4, we relax the quasimonotonicity condition for the reaction nonlinearities but further restrict, as a cost of the above relaxation, the upper solution as the initial function of the iteration. Using some nonstandard ordering of the profile set we prove that the monotone iteration established in Section 3 remains valid. Thus the main results in Sections 3 and 4 enable us to establish the existence of traveling wave fronts of a reaction-diffusion system by choosing an appropriate pair of lower and upper solutions of the corresponding wave equation. In Section 5, the main results are illustrated by and applied to the Hutchinson equation and the Belousov–Zhabotinskii reaction model for which the required pair of the lower and upper solutions are analytically constructed and thus the existence of traveling wave fronts can be obtained.

2. PRELIMINARIES

Consider the following system of reaction-diffusion equations with time delay,

$$\frac{\partial u(t, x)}{\partial t} = D\frac{\partial^2 u(t, x)}{\partial x^2} + f(u_t(x)) \tag{2.1}$$

where $t \geqslant 0$, $x \in \mathbb{R}$, $u \in \mathbb{R}^n$, $D = diag(d_1,..., d_n)$ with $d_i > 0$, $i = 1,..., n$, f: $C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is continuous, and $u_t(x)$ is an element in $C([-\tau, 0]; \mathbb{R}^n)$ parameterized by $x \in \mathbb{R}$ and given by

$$u_t(x)(s) = u(t+s, x), \qquad s \in [-\tau, 0], \quad t \geqslant 0, \quad x \in \mathbb{R} \tag{2.2}$$

A traveling wave solution of (2.1) is a solution of the form $u(t, x) = \phi(x+ct)$, where $\phi \in C^2(\mathbb{R}; \mathbb{R}^n)$ and $c > 0$ is a constant corresponding to the wave speed. Substituting $u(t, x) = \phi(x+ct)$ into (2.1) and letting $s = x+ct$, which is called a traveling coordinate, one knows that ϕ must be a solution of the following system of functional differential equations,

$$D\phi''(s) - c\phi'(s) + f_c(\phi_s) = 0, \qquad s \in \mathbb{R} \tag{2.3}$$

where f_c: $X_c = C([-c\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is defined by

$$\begin{cases} f_c(\psi) = f(\psi^c) \\ \psi^c(s) = \psi(cs), \qquad s \in [-\tau, 0] \end{cases} \tag{2.4}$$

If for some $c > 0$, (2.3) has a monotone solution 0 defined on $\mathbb{R}$ such that

$$\begin{cases} \lim_{s \to -\infty} \phi(s) = u_- \\ \lim_{s \to +\infty} \phi(s) = u_+ \end{cases} \tag{2.5}$$

exist, then $u(tx,) = \phi(x + ct)$ is called a wave front of (2.1) with speed c.

In what follows, we explore the existence of solutions of (2.3) subject to (2.5). Note that the wave speed is unknown and needs to be determined while solving (2.3) and (2.5). So, (2.3) subject to (2.5) is in fact an eigenvalue problem.

In the remainder of this paper, we use the usual notations for the standard ordering in $\mathbb{R}^n$. That is, for $u = (u_1,..., u_n)^T$ and $v = (v_1,..., v_n)^T \in \mathbb{R}^n$, we denote $u \leqslant v$ if $u_i \leqslant v_i$, $i = 1,..., n$, and $u < v$ if $u \leqslant v$ but $u \neq v$.

Proposition 2.1. *If (2.3) and (2.5) has a monotone solution, then* $f_c(\hat{u}_-) = f_c(\hat{u}_+) = \mathbf{0}$, *where for* $u \in \mathbb{R}^n$, $\hat{u}$ *denotes the constant vector function on* $[-c\tau, 0]$ *taking the value* u.

In order to prove this proposition, we need the following two lemmas.

Lemma 2.2 (Fluctuation Lemma). *Let* x: $\mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function. If*

$$\liminf_{t \to \infty} x(t) < \limsup_{t \to \infty} x(t)$$

then there are sequences $\{t_n\}$ *and* $\{s_n\}$ *with* $\lim_{n \to \infty} t_n = \infty$ *and* $\lim_{n \to \infty} s_n = \infty$ *such that*

$$\lim_{n \to \infty} x(t_n) = \limsup_{t \to \infty} x(t) \qquad \text{and} \qquad x'(t_n) = 0$$

$$\lim_{n \to \infty} x(s_n) = \liminf_{t \to \infty} x(t) \qquad \text{and} \qquad x'(t_n) = 0$$

Lemma 2.3. *Let* $a \in (-\infty, \infty)$ *and* $x\colon [a, \infty) \to \mathbb{R}$ *be a differentiable function. If* $\lim_{t\to\infty} x(t)$ *exists (finite) and the derivative function* $x'(t)$ *is uniformly continuous on* $[a, \infty)$, *then* $\lim_{t\to\infty} x'(t) = 0$.

For a proof of Lemma 2.2, see Hirsch *et al.* (1985); and for a proof of Lemma 2.3, see Barbalat (1959) or Gopalsamy (1992).

Proof of Proposition 2.1. Let $\phi\colon \mathbb{R} \to \mathbb{R}^n$ be a solution of (2.3) and (2.5). Then for every $t \in \mathbb{R}$, we have $\lim_{t\to\infty} \phi_t = \hat{u}_+$ and $\lim_{t\to-\infty} \phi_t = \hat{u}_-$. By the continuity of f (and hence, of f_c), we have

$$\lim_{t\to\infty} f_c(\phi_t) = f_c(\hat{u}_+), \qquad \lim_{t\to-\infty} f_c(\phi_t) = f_c(\hat{u}_-) \tag{2.6}$$

Now, we fix $i \in \{1, \ldots, n\}$ and look at the ith equation in (2.3),

$$d_i \phi_i''(t) - c\phi_i'(t) + (f_c(\phi))_i = 0, \qquad t \in \mathbb{R} \tag{2.7}$$

We first show that $\lim_{t\to\infty} \phi_i'(t)$ exists. Let $L_i = \lim_{t\to\infty} \sup \phi_i'(t)$ and $l_i = \lim_{t\to\infty} \inf \phi_i'(t)$. For the sake of contradiction, we assume that $l_i < L_i$. Then. by Lemma 2.2, there exist sequences t_n and s_n with $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \infty$ such that

$$\begin{aligned} &\lim_{n\to\infty} \phi_i'(t_n) = L_i, \qquad \phi_i''(t_n) = 0 \\ &\lim_{n\to\infty} \phi_i'(s_n) = l_i, \qquad \phi_i''(s_n) = 0 \end{aligned} \tag{2.8}$$

It turns out from (2.6)–(2.8) that

$$\begin{aligned} 0 - cL_i + (f_c(\hat{u}_+))_i &= 0 \\ 0 - cl_i + (f_c(\hat{u}_+))_i &= 0 \end{aligned} \tag{2.9}$$

Now we have two cases to consider.

Case 1. $c \neq 0$: (2.9) yields $c(L_i - L_i) = 0$ and thus $L_i = l_i$. This is a contradiction to $l_i < L_i$.

Case 2. $c = 0$: In this case, (2.9) implies that $(f_c(\hat{u}_+))_i = 0$. On the other hand, (2.7) becomes

$$d_i \phi_i''(t) + (f_c(\phi_t))_i = 0, \qquad t \in \mathbb{R} \tag{2.10}$$

which implies that $\lim_{t\to\infty} \phi_i''(t) = (-1/d_i)(f_c(\hat{u}_+))_i = 0$. Thus, $\phi_i'(t)$ is uniformly continuous on $[0, \infty)$, and hence (by Lemma 2.3), $\lim_{t\to\infty} \phi_i'(t) = 0$ since $\lim_{t\to\infty} \phi_i(t) = (u_+)_i$ exists. This again leads to the contradictions to $l_i < L_i$.

So, we must have $l_i = L_i$, i.e., $\lim_{t\to\infty} \phi'_i(t)$ exists. But $\lim_{t\to\infty} \phi_i(t)$ also exists (hence $\phi_i(t)$ is bounded on $[0, \infty)$). This claims that $\lim_{t\to\infty} \phi'_i(t)$ must be zero. Now we take the limit as $t \to \infty$ in (2.7) and find that $\lim_{t\to\infty} \phi''(t) = \lim_{t\to\infty} (c/d_i)\, \phi'_i(t) - \lim_{t\to\infty} (1/d_i)(f_c(\phi_t))_i = 0 - (1/d_i)(f_c(\hat{u}_+))_i$ also exists. Again, by the boundedness of $\phi'_i(t)$, we know that $\lim_{t\to\infty} \phi''_i(t) = 0$. Therefore, we obtain $(f_c(\hat{u}_+))_i = 0$ by letting $t \to \infty$ in (2.7).

Noting that $i \in \{1,..., n\}$ is arbitrary, we have actually proved that $f_c(\hat{u}_+) = \mathbf{0}$. In a similar way, we can prove that $f_c(\hat{u}_-) = \mathbf{0}$. This completes the proof of the proposition.

It is now natural to assume, as far as looking for monotone solutions of (2.3) and (2.5) is concerned, that f has at least two zeros, u_- and u_+. Without loss of generality, we can assume $u_- = \mathbf{0}$ and $u_+ = K > \mathbf{0}$. More precisely, throughout the remainder of this paper, we assume the following:

(A1) $f(\hat{\mathbf{0}}) = f(\hat{K}) = \mathbf{0}$ and $f(\hat{u}) \neq \mathbf{0}$ for $u \in \mathbb{R}^n$ with $\mathbf{0} < u < K$.

Obviously we should replace (2.5) with

$$\begin{cases} \lim\limits_{t\to-\infty} \phi(t) = \mathbf{0} \\ \lim\limits_{t\to+\infty} \phi(t) = K \end{cases} \tag{2.11}$$

3. THE EXISTENCE OF WAVE FRONTS: MONOTONE DELAYED REACTION

In this section, we explore the existence of wave fronts of (2.1) where the reaction term f is monotone with respect to the delayed arguments. In other words, we assume, in addition to (A1) in Section 2, the following quasimonotonicity condition:

(A2) There exists a matrix $\beta = diag(\beta_1,..., \beta_n)$ with $\beta_i \geqslant 0$ such that

$$f_c(\phi) - f_c(\psi) + \beta[\phi(0) - \psi(0)] > \mathbf{0} \qquad \text{for} \quad \phi, \psi \in X_c,$$

$$\text{with} \quad \mathbf{0} \leqslant \psi(s) \leqslant \phi(s) \leqslant K, \qquad s \in [-c\tau, 0]$$

As mentioned in the Introduction, we develop iteration scheme to approach a solution of (2.3) and (2.11). To start with, we define the following profile set:

$$\Gamma = \left\{ \phi \in C(\mathbb{R}; \mathbb{R}^n); \quad \begin{array}{ll} \text{(i)} & \phi \text{ is nondecreasing in } \mathbb{R} \\ \text{(ii)} & \lim_{t\to-\infty} \phi(t) = \mathbf{0}, \lim_{t\to\infty} \phi(t) = K \end{array} \right\} \tag{3.1}$$

We also define H: $C(\mathbb{R}; \mathbb{R}^n) \to C(\mathbb{R}; \mathbb{R}^n)$ by

$$H(\phi)(t) = f_c(\phi_t) + \beta\phi(t), \qquad \phi \in C(\mathbb{R}, \mathbb{R}^n), \quad t \in \mathbb{R} \tag{3.2}$$

Now we explore some basic properties of H:

Lemma 3.1. *Assume that (A1) and (A2) hold. Then for any $\phi \in \Gamma$, we have that*

(i) $H(\phi)(t) \geqslant 0$, $t \in \mathbb{R}$;

(ii) $H(\phi)(t)$ *is nondecreasing in* $t \in \mathbb{R}$;

(iii) $H(\psi)(t) \leqslant H(\phi)(t)$, *for* $t \in \mathbb{R}$, *if* $\psi \in C(\mathbb{R}; \mathbb{R}^n)$ *is given so that* $\mathbf{0} \leqslant \psi(t) \leqslant \phi(t) \leqslant K$ *for* $t \in \mathbb{R}$.

Proof. (i) is a direct consequence of (A1)–(A2) as $f_c(\mathbf{0}) = \mathbf{0}$. (iii) follows immediately from (A2). To prove (ii), we let $t \in \mathbb{R}$ and $s > 0$ be given. Then

$$\mathbf{0} \leqslant \phi_t(\theta) \leqslant \phi_{t+s}(\theta) \leqslant K$$

and hence

$$\begin{aligned} H(\phi)(t+s) - H(\phi)(t) &= f_c(\phi_{t+s}) - f_x(\phi_t) + \beta[\phi(t+s) - \phi(t)] \\ &= f_c(\phi_{t+s}) - f_c(\phi_t) + \beta[\phi_{t+s}(0) - \phi_t(0)] \\ &\geqslant 0 \end{aligned}$$

by (A2). This completes the proof.

We start our iteration with an upper solution of (2.3) defined as follows.

Definition 3.2. A continuous function ρ: $\mathbb{R} \to \mathbb{R}^n$ is called an upper solution of (2.3) if ρ' and ρ'' exist almost everywhere and they are essentially bounded on $\mathbb{R}$, and if ρ satisfies

$$D\rho''(t) - c\rho'(t) + f_c(\rho_t) \leqslant \mathbf{0}, \qquad \text{a.e. on } \mathbb{R} \tag{3.3}$$

A lower solution of (2.3) is defined in a similar way by reversing the inequality in (3.3).

In what follows, we assume that an upper solution $\bar{\rho} \in \Gamma$ and a lower solution of $\underline{\rho}$ (which is not necessarily in Γ) of (2.3) are given so that

(H1) $\mathbf{0} \leqslant \underline{\rho}(t) \leqslant \bar{\rho}(t) \leqslant K$, $t \in \mathbb{R}$;

(H2) $\underline{\rho}(t) \not\equiv \mathbf{0}$, $t \in \mathbb{R}$.

Our first iteration involves the following linear nonhomogeneous system of ordinary differential equations

$$cx_1'(t) = Dx_1''(t) - \beta x_1(t) + H(\bar{\rho})(t), \qquad t \in \mathbb{R} \tag{3.4}$$

Among all solutions of (3.4), we choose a special one and explore its properties as below.

Lemma 3.3. *Let*

$$\lambda_{1i} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, \qquad \lambda_{2i} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i} \tag{3.5}$$

and define

$$x_{1i}(t) = \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} H_i(\bar{\rho})(s)\, ds + \int_t^{\infty} e^{\lambda_{2i}(t-s)} H_i(\bar{\rho})(s)\, ds\right] \tag{3.6}$$

for $t \in \mathbb{R}$ and $i = 1, \ldots, n$. Then we have that

(i) $x_1(t) \triangleq (x_{11}(t), \ldots, x_{1n}(t))^T$ *solves (3.4);*

(ii) $x_1 \in \Gamma$;

(iii) $\underline{\rho}(t) \leqslant x_1(t) \leqslant \bar{\rho}(t)$, $t \in \mathbb{R}$;

(iv) x_1 *is an upper solution of (2.3).*

Proof. The proof of (i) is a direct verification and is thus omitted here. For (ii), we first show that $x_1(t)$ satisfies the two limiting conditions for Γ. Indeed, applying L. Hopital's rule to (3.6) and using (A1), (3.2), and the fact that $\bar{\rho} \in \Gamma$, we get

$$\begin{aligned}
\lim_{t\to-\infty} x_{1i}(t) &= \lim_{t\to-\infty} \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \\
&\quad \times \left[\frac{\int_{-\infty}^{t} e^{-\lambda_{1i}s} H_i(\bar{\rho})(s)\, ds}{e^{-\lambda_{1i}t}} + \frac{\int_t^{\infty} e^{-\lambda_{2i}s} H_i(\bar{\rho})(s)\, ds}{e^{-\lambda_{2i}t}}\right] \\
&= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \lim_{t\to-\infty} \left[\frac{H_i(\bar{\rho})(t)}{-\lambda_{1i}} - \frac{H_i(\bar{\rho})(t)}{-\lambda_{2i}}\right] \\
&= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})}\left[\frac{0}{-\lambda_{1i}} + \frac{0}{\lambda_{2i}}\right] = 0, \qquad i = 1, \ldots, n
\end{aligned}$$

and

$$\lim_{t\to+\infty} x_{1i}(t) = \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})} \lim_{t\to-\infty} \left[\frac{H_i(\bar{\rho})(t)}{-\lambda_{1i}} - \frac{H_i(\bar{\rho})(t)}{\lambda_{2i}} \right]$$

$$= \frac{-1}{d_i(\lambda_{2i}-\lambda_{1i})} \left[\frac{\beta_i K_i}{\lambda_{1i}} - \frac{\beta_i K_i}{-\lambda_{2i}} \right]$$

$$= \frac{\beta_i K_i}{-\lambda_{1i}\lambda_{2i} d_i} = \frac{\beta_i K_i}{\beta_i} = K_i, \qquad i = 1,..., n$$

Thus, we have $\lim_{t\to+\infty} x_1(t) = (K_1,..., K_n)^T$ and $\lim_{t\to-\infty} x_1(t) = (0,..., 0)^T = \mathbf{0}$.

Next we prove that each component of x_1 is nondecreasing in $\mathbb{R}$. To this end, we let $t \in \mathbb{R}$ and $s > 0$ be given. Then

$$\begin{aligned}
&x_{1i}(t+s) - x_{1i}(t) \\
&\quad = \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})} \left[\int_{-\infty}^{t+s} e^{\lambda_{1i}(t+s-\theta)} H_i(\bar{\rho})(\theta)\, d\theta + \int_{t+s}^{\infty} e^{\lambda_{2i}(t+s-\theta)} H_i(\bar{\rho})(\theta)\, d\theta \right] \\
&\qquad - \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-\theta)} H_i(\bar{\rho})(\theta)\, d\theta + \int_{t}^{\infty} e^{\lambda_{2i}(t-\theta)} H_i(\bar{\rho})(\theta)\, d\theta \right] \\
&\quad = \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})} \int_{-\infty}^{t} e^{\lambda_{1i}(t-\theta)} [H_i(\bar{\rho})(\theta+s) - H_i(\bar{\rho})(\theta)\, d\theta] \\
&\qquad + \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})} \int_{t}^{+\infty} e^{\lambda_{2i}(t-\theta)} [H_i(\bar{\rho})(\theta+s) - H_i(\bar{\rho})(\theta)\, d\theta] \\
&\quad \geqslant 0 \qquad [\text{by Lemma 3.1 (ii)}]
\end{aligned}$$

This completes the proof of (ii).

To justify (iii), we let $w_i(t) = x_{1i}(t) - \bar{\rho}_i(t)$, $t \in \mathbb{R}$, $i = 1,..., n$. Combining (3.3) with (3.4), we have

$$c w_i'(t) \leqslant d_i w_i''(t) - \beta_i w_i(t), \qquad t \in \mathbb{R}, \quad i = 1,..., n \tag{3.7}$$

Denote $r_i(t) = c w_i'(t) - d_i w_i''(t) + \beta_i w_i(t)$, $t \in \mathbb{R}$, $i = 1,..., n$. Then r_i is essentially bounded and nonpositive on $\mathbb{R}$, $i = 1,..., n$. Now from

$$d_i w_i''(t) - c w_i'(t) - \beta_i w_i(t) = -r_i(t), \qquad t \in \mathbb{R}, \quad i = 1,..., n \tag{3.8}$$

and the fundamental theory of second-order linear ordinary differential equations. we get

$$w_i(t) = c_1 e^{\lambda_{1i} t} + c_2 e^{\lambda_{2i} t} + \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})}$$
$$\times \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} r_i(s)\, ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)} r_i(s)\, ds \right] \tag{3.9}$$

for $t \in \mathbb{R}$, $i = 1,..., n$, where c_1 and c_2 are constants. Note that $\lambda_{1i} < 0$, $\lambda_{2i} > 0$, $\lim_{t \to -\infty} w_i(t) = \lim_{t \to -\infty} x_{1i}(t) - \lim_{t \to -\infty} \bar{\rho}_i(t) = 0 - 0 = 0$ and $\lim_{t \to +\infty} w_i(t) = \lim_{t \to +\infty} x_{1i}(t) - \lim_{t \to +\infty} \bar{\rho}_i(t) = K_i - K_i = 0$. Therefore, we must have $c_1 = c_2 = 0$. Consequently,

$$\begin{aligned} w_i(t) &= x_{1i}(t) - \bar{\rho}_i(t) \\ &= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} r_i(s)\, ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)} r_i(s)\, ds \right] \\ &\leqslant 0, \qquad t \in \mathbb{R}, \quad i = 1,..., n \end{aligned}$$

since $r_i(t) \leqslant 0$, $t \in \mathbb{R}$, $i = 1,..., n$. This proves that $x_1(t) \leqslant \bar{\rho}(t)$, $t \in \mathbb{R}$.

In a similar way, we can prove that $\underline{\rho}(t) \leqslant x_1(t)$ for $t \in \mathbb{R}$, by using (A2). This justifies (iii).

We now show that x_1 is an upper solution of (2.3). In fact, we have

$$\begin{aligned} cx_1'(t) &= Dx_1''(t) - \beta x_1(t) + H(\bar{\rho})(t) \\ &= Dx_1''(t) + f(x_{1t}) + [H(\bar{\rho})(t) - H(x_1)(t)] \\ &\geqslant Dx_1''(t) + f(x_{1t}), \qquad t \in \mathbb{R} \end{aligned}$$

by Lemma 3.1(iii). This completes the proof.

As x_1 is also an upper solution, we can replace $\bar{\rho}$ with x_1 and consider the following linear nonhomogeneous system of ordinary differential equations:

$$cx_2'(t) = Dx_2''(t) - \beta x_2(t) + H(x_1)(t), \qquad t \in \mathbb{R} \tag{3.10}$$

In general, we consider the following iteration scheme:

$$\begin{cases} cx_m'(t) = Dx_m''(t) - \beta x_m(t) + H(x_{m-1})(t), & t \in \mathbb{R}, \quad m = 1, 2,... \\ x_0 = \bar{\rho} \end{cases} \tag{3.11}$$

Then we can construct, by repeating the above argument for x_1, a sequence $\{x_m\}_{m=1}^{\infty}$ of vector functions defined on $\mathbb{R}$, where $x_m(t)=(x_{m1}(t),\dots, x_{mn}(t))^T$ with

$$\begin{cases} x_{mi}(t)=\dfrac{1}{d_i(\lambda_{2i}-\lambda_{1i})} \\ \qquad \times\left[\displaystyle\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}H_i(x_{m-1})(s)\,ds+\int_{t}^{\infty} e^{\lambda_{2i}(t-s)}H_i(x_{m-1})(s)\,ds\right] \\ x_{0i}(t)=\bar{\rho}_i(t), \qquad t\in\mathbb{R} \end{cases} \tag{3.12}$$

where $t\in\mathbb{R}$, $i=1,\dots, n$, and $m=1, 2,\dots$. We can inductively establish the following.

Lemma 3.4. *$x_m(t)$ defined by (3.12) satisfies*

(i) *x_m solves (3.11) on $\mathbb{R}$ for $m=1, 2,\dots$;*
(ii) *$x_m\in\Gamma$;*
(iii) *$\underline{\rho}(t)\leqslant x_m(t)\leqslant x_{m-1}(t)\leqslant\bar{\rho}(t)$, $t\in\mathbb{R}$;*
(iv) *each x_m is an upper solution of (2.3).*

From (iii) in the above lemma, we know that $x(t)=\lim_{m\to\infty} x_m(t)$ exists and satisfies $\underline{\rho}(t)\leqslant x(t)\leqslant\bar{\rho}(t)$ for $t\in\mathbb{R}$. Moreover, $x(t)$ is nondecreasing in $t\in\mathbb{R}$ since $x_m(t)$ is for each fixed $m=1, 2,\dots$. We next prove that $x(t)$ solves (2.3) and (2.11).

Proposition 3.5. *$x(t)=\lim_{m\to\infty} x_m(t)$ is a solution of (2.3) and (2.6).*

Proof. By the continuity of f, Lebesque's dominated convergence theorem, and (3.12), we have

$$\begin{aligned} x_i(t)&=\lim_{m\to\infty} x_{mi}(t) \\ &=\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\lim_{m\to\infty}\left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}H_i(x_{m-1})(s)\,ds\right. \\ &\quad\left.+\int_{t}^{\infty} e^{\lambda_{2i}(t-s)}H_i(x_{m-1})(s)\,ds\right] \\ &=\frac{1}{d_i(\lambda_{1i}-\lambda_{1i})}\left[e^{\lambda_{1i}t}\int_{-\infty}^{t} e^{-\lambda_{1i}s}H_i(x)(s)\,ds+e^{\lambda_{2i}t}\int_{t}^{\infty} e^{-\lambda_{2i}s}H_i(x)(s)\,ds\right] \end{aligned} \tag{3.13}$$

for $t \in \mathbb{R}$, $i = 1, \ldots, n$. Direct calculation shows that

$$x_i'(t) = \lambda_{1i} e^{\lambda_{1i} t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds \\ + \lambda_{2i} e^{\lambda_{2i} t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds \tag{3.14}$$

and

$$\begin{aligned} x_i''(t) &= \lambda_{1i}^2 e^{\lambda_{1i} t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds \\ &\quad + \lambda_{2i}^2 e^{\lambda_{2i} t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds + \frac{H_i(x)(t)}{d_i(\lambda_{2i} - \lambda_{1i})} [\lambda_{1i} - \lambda_{2i}] \\ &= \lambda_{1i}^2 e^{\lambda_{1i} t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds \\ &\quad + \lambda_{2i}^2 e^{\lambda_{2i} t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds - \frac{1}{d_i} H_i(x)(t) \end{aligned} \tag{3.15}$$

Thus

$$\begin{aligned} &d_i x_i''(t) - c x_i'(t) - \beta_i x_i(t) \\ &\quad = (d_i \lambda_{1i}^2 - c\lambda_{1i} - \beta_i)\, e^{\lambda_{1i} t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds \\ &\qquad + (d_i \lambda_{2i}^2 - c\lambda_{2i} - \beta_i)\, e^{\lambda_{2i} t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} H_i(x)(s)\, ds - H_i(x)(t) \\ &\quad = -H_i(x)(t), \qquad t \in \mathbb{R}, \quad i = 1, \ldots, n \end{aligned}$$

Therefore

$$Dx''(t) - cx'(t) - \beta x(t) = -H(x)(t) = -f_c(x_t) - \beta x(t), \qquad t \in \mathbb{R}$$

that is

$$cx'(t) = Dx''(t) + f_c(x_t), \qquad t \in \mathbb{R}$$

So, $x(t)$ solves (2.3).

From Lemma 3.4(iii) and $\lim_{t \to -\infty} \bar{\rho}(t) = \mathbf{0}$, we immediately have $\lim_{t \to -\infty} x(t) = \mathbf{0}$. On the other hand, $x(t)$ is nondecreasing and bounded from above by K. Hence $\lim_{t\to+\infty} x(t) = K_0 = (K_{01},..., K_{0n})$ exists and $\sup_{t\in\mathbb{R}} \underline{\rho}_i(t) \leqslant K_{0i} \leqslant K_i$, $i=1,..., n$. Recall that we have assumed that $\mathbf{0} \leqslant \underline{\rho}(t) \not\equiv \mathbf{0}$, $t\in\mathbb{R}$. This implies that $K_0 \in (0, K]$. Applying l'Hospital's rule to (3.13) and using the continuity of f, we get

$$\begin{aligned} K_{0i} = \lim_{t\to+\infty} x_i(t) &= \lim_{t\to+\infty} \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\left[\frac{H_i(x)(t)}{-\lambda_{1i}}+\frac{H_i(x)(t)}{\lambda_{2i}}\right] \\ &= \frac{f_i(\hat{K}_0)+\beta_i K_{0i}}{-d_i\lambda_{2i}\lambda_{1i}} \\ &= \frac{f_i(\hat{K}_0)+\beta_i K_{0i}}{\beta_i} \\ &= \frac{f_i(\hat{K}_0)}{\beta_i}+K_{0i}, \qquad i=1,..., n \end{aligned} \tag{3.16}$$

This leads to $f_i(\hat{K}_0)=0$, $i=1,..., n$, i.e, $f(\hat{K}_0)=\mathbf{0}$. Now, by (A1), we finally arrive at $K_0 = K$. So $x(t)$ also satisfies (2.11). This completes the proof.

Summarizing the above lemmas and Proposition 3.5, we have proved the following.

Theorem 3.6. *Assume that (A1) and (A2) hold. Suppose that (2.3) has an upper solution $\bar{\rho}$ in Γ and a lower solution $\underline{\rho}$ (which is not necessarily in Γ) with $\mathbf{0} \leqslant \underline{\rho}(t) \leqslant \bar{\rho}(t) \leqslant K$ and $\underline{\rho}(t) \not\equiv \mathbf{0}$ in $\mathbb{R}$. Then (2.3) and (2.11) have a solution. That is, (2.1) has a traveling wave front solution.*

4. EXISTENCE OF WAVE FRONTS: NONMONOTONE DELAYED REACTION

In this section, we relax the quasimonotonicity condition (A2) in Section 3. As a cost of this relaxation, we have to impose more restrictions on the upper and lower solutions employed as initial iteration. More precisely. we relax (A2) as

(A2)* There exists a matrix $\beta = diag(\beta_1,..., \beta_n)$ with $\beta_i \geqslant 0$, $i=1,..., n$, such that

$$f_c(\phi) - f_c(\psi) + \beta[\phi(0) - \psi(0)] \geqslant 0$$

for $\phi, \psi \in X_c$ with (i) $\mathbf{0} \leqslant \psi(s) \leqslant \phi(s) \leqslant K$ for $s\in[-c\tau, 0]$; (ii) $e^{\beta s}[\phi(s)-\psi(s)]$ nondecreasing in $s\in[-c\tau, 0]$, and we look for wave front solutions of (2.1) in the following profile set:

$$\Gamma^* = \left\{ \phi \in C(\mathbb{R}; \mathbb{R}^n); \quad \begin{array}{rl} \text{(i)} & \phi \text{ is nondecreasing in } \mathbb{R} \\ \text{(ii)} & \lim_{t \to -\infty} \phi(t) = \mathbf{0}, \lim_{t \to +\infty} \phi(t) = K \\ \text{(iii)} & e^{\beta t}[\phi(t+s) - \phi(t)] \text{ is nondecreasing in} \\ & t \in \mathbb{R} \text{ for every } s > 0 \end{array} \right\}$$

Condition (A2)* is motivated by the nonstandard ordering of a phase space introduced by Smith and Thieme (1990, 1991) in order to obtain the (strong) order-preserving property of the solution semiflows defined by noncooperative functional differential equations, and in order to apply the powerful theory of monotone dynamical systems.

Let H: $(C(\mathbb{R}; \mathbb{R}^n) \to C(\mathbb{R}; \mathbb{R}^n)$ be defined by (3.2). We have the following analogue of Lemma 3.1.

Lemma 4.1. *Assume that (A1) and (A2)* hold. Then for any $\phi \in \Gamma^*$, we have that*

(i) $H(\phi)(t) \geqslant 0, \ t \in \mathbb{R}$;

(ii) $H(\phi)(t)$ *is nondecreasing in* $t \in \mathbb{R}$;

(iii) $H(\psi)(t) \leqslant H(\phi)(t)$ *for* $t \in \mathbb{R}$ *if* $\psi \in C(\mathbb{R}; \mathbb{R}^n)$ *satisfies that* $\mathbf{0} \leqslant \psi(t) \leqslant \phi(t) \leqslant K$ *and that* $e^{\beta t}[\phi(t) - \psi(t)]$ *is nondecreasing in* $t \in \mathbb{R}$.

Proof. (i) and (iii) follow directly from (A1) and (A2)*, using the same argument as for (i) and (iii) of Lemma 3.1. To prove (ii), we let $s > 0$ be given. Note that

$$e^{\beta\theta}[\phi_{t+s}(\theta) - \phi_t(\theta)] = e^{-\beta t} e^{\beta(t+\theta)}[\phi(t+\theta+s) - \phi(t+\theta)]$$

is nondecreasing in $\theta \in [-c\tau, 0]$ since $\phi \in \Gamma^*$. This implies that

$$\begin{aligned} & H(\phi)(t+s) - H(\phi)(t) \\ & \qquad = f_c(\phi_{t+s}) - f_c(\phi_t) + \beta[\phi_{t+s}(0) - \phi_t(0)] \geqslant \mathbf{0} \qquad t \in \mathbb{R} \end{aligned}$$

Thus, $H(\phi)(t)$ is nondecreasing in $t \in \mathbb{R}$. This completes the proof.

Parallel to Section 3, we assume that an upper solution $\bar{\phi} \in \Gamma^*$ and a lower solution $\underline{\phi}$ (which is not necessarily in Γ^*) are given and satisfy

(H1)* $\mathbf{0} \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant K, \ t \in \mathbb{R}$;

(H2)* $\underline{\phi}(t) \not\equiv \mathbf{0}$;

(H3)* $e^{\beta t}[\bar{\phi}(t) - \underline{\phi}(t)]$ is nondecreasing in $\mathbb{R}$.

Starting with $\bar{\phi}$, we consider

$$cy_1'(t) = Dy_1''(t) - \beta y_1(t) + H(\bar{\phi})(t), \qquad t \in \mathbb{R} \tag{4.1}$$

which is a linear nonhomogeneous system of ordinary differential equations. Among all the solutions we again pick up, just as in Section 3, a particular one. The definition and some properties of this particular solution are given in the following.

Lemma 4.2. *Let λ_{1i} and λ_{2i} be given by (3.5). Let $y_1(t) = (y_{11}(t),..., y_{1n}(t))^T$ be defined by*

$$y_{1i}(t) = \int_{-\infty}^{t} \frac{e^{\lambda_{1i}(t-s)}}{d_i(\lambda_{2i}-\lambda_{1i})} H_i(\bar{\phi})(s)\, ds + \int_{t}^{+\infty} \frac{e^{\lambda_{2i}(t-s)}}{d_i(\lambda_{2i}-\lambda_{1i})} H_i(\bar{\phi})(s)\, ds, \qquad t \in \mathbb{R}, \quad i = 1,..., n \tag{4.2}$$

Then we have that

(i) $y_1(t)$ solves (4.1);

(ii) if $c > 1 - \min\{\beta_i d_i;\, i = 1,..., n\}$, then $y_1 \in \Gamma^*$;

(iii) $\underline{\phi}(t) \leqslant y_1(t) \leqslant \bar{\phi}(t)$ for $t \in \mathbb{R}$.

Proof. The proof of (i) and the verification of the fact that $y_1(t)$ satisfies (i) and (ii) in Γ^* are exactly that same as those in the argument of Lemma 3.3. We need to show only that y_1 also satisfies (iii) in Γ^*, that is, $e^{\beta t}[y_1(t+s) - y_1(t)]$ is nondecreasing in $t \in \mathbb{R}$ for any given $s > 0$. To this end, let $s > 0$ be given. Then we have that

$$\begin{aligned} &e^{\beta_i t}[y_{1i}(t+s) - y_{1i}(t)] \\ &= e^{(\beta_i + \lambda_{1i})\, t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i}\theta}}{d_i(\lambda_{2i}-\lambda_{1i})} [H(\bar{\phi})(+s) - H(\bar{\phi})(\theta)]\, d\theta \\ &\quad + e^{(\beta_i + \lambda_{2i})\, t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i}\theta}}{d_i(\lambda_{2i}-\lambda_{1i})} [H(\bar{\phi})(\theta+s) - H(\bar{\phi})(\theta)]\, d\theta, \;\; t \in \mathbb{R}, \;\; i = 1,..., n \end{aligned}$$

Note that

$$\beta_i + \lambda_{2i} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i} + \beta_i > 0, \qquad i = 1,..., n$$

and

$$\beta_i + \lambda_{1i} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i} + \beta_i$$

$$= \frac{2\beta_i (c - 1 + d_i \beta_i)}{(c + 2\beta_i d_i) + \sqrt{c^2 + 4\beta_i d_i}} > 0, \qquad i = 1,..., n$$

since $c > 1 - \min\{\beta_i d_i; i = 1,..., n\}$. Thus, by Lemma 3.1 and the chain rule, we get that

$$\frac{d}{dt}\{e^{\beta_i t}[y_{1i}(t+s) - y_{1i}(t)]\}$$

$$= (\beta_i + \lambda_{1i})\, e^{(\beta_i + \lambda_{1i})\, t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i}\theta}}{d_i(\lambda_{2i} - \lambda_{1i})} [H(\bar{\phi})(\theta + s) - H(\bar{\phi})(\theta)]\, d\theta$$

$$+ (\beta_i + \lambda_2 i)\, e^{(\beta_i + \lambda_{2i})\, t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i}\theta}}{d_i(\lambda_{2i} - \lambda_{1i})} [H(\bar{\phi})(\theta + s) - H(\bar{\phi})(\theta)]\, d\theta$$

$$\geqslant 0, \qquad t \in \mathbb{R}, \quad i = 1,..., n$$

This proves (ii). The justification of (iii) is the same as that for (iii) of Lemma 3.3 and is omitted.

Lemma 4.3. *Let y_1 be as in Lemma 4.2. If $c > 1 - \min\{\beta_i d_i;\ i = 1,..., n\}$, then $e^{\beta t}[y_1(t) - \underline{\phi}(t)]$ is nondecreasing in $t \in \mathbb{R}$.*

Proof. Let $u(t) = y_1(t) - \underline{\phi}(t)$, $t \in \mathbb{R}$. Then, by Lemma 4.1, we get that

$$cu'(t) = cy_1'(t) - c\underline{\phi}'(t)$$

$$\geqslant [Dy_1''(t) - \beta y_1(t) + H(\bar{\phi})(t)] - [D\underline{\phi}''(t) - \beta\underline{\phi}(t) + H(\underline{\phi})(t)]$$

$$= Du''(t) - \beta u(t) + [H(\bar{\phi})(t) - H(\underline{\phi})(t)]$$

$$\geqslant Du''(t) - \beta u(t), \qquad t \in \mathbb{R} \tag{4.3}$$

Denote $h_i(t) = cu_i'(t) - d_i u_i''(t) + \beta_i u_i(t)$, $t \in \mathbb{R}$, $i = 1,..., n$. Then, $h_i(t) \geqslant 0$, $t \in \mathbb{R}$, $i = 1,..., n$, by (3.3). From

$$d_i u_i''(t) - cu_i'(t) - \beta_i u_i(t) = -h_i(t), \qquad t \in \mathbb{R}, \quad i = 1,..., n \tag{4.4}$$

and the elementary theory of second-order linear ordinary differential equations, we know that

$$u_i(t) = c_1 e^{\lambda_{1i} t} + c_2 e^{\lambda_{2i} t} + \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \times \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} h_i(s)\, ds + \int_{t}^{-\infty} e^{\lambda_{2i}(t-s)} h_i(s)\, ds \right] \tag{4.5}$$

for $t \in \mathbb{R}$, $i = 1, \ldots, n$. The boundedness of u_i and the fact that $\lambda_{1i} < 0$ and $\lambda_{2i} > 0$ imply that $c_1 = c_2 = 0$. Thus

$$\begin{aligned} y_{1i}(t) - \underline{\phi}(t) &= u_i(t) \\ &= \frac{1}{d_i(\lambda_{2i} - \lambda_{1i})} \left[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} h_i(s)\, ds + \int_{t}^{+\infty} e^{\lambda_{2i}(t-s)} h_i(s)\, ds \right] \\ &\geqslant 0, \qquad t \in \mathbb{R}, \quad i = 1, \ldots, n \end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned} &e^{\beta_i t}[y_{1i}(t) - \underline{\phi}_i(t)] \\ &= e^{(\beta_i + \lambda_{1i})t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(s)\, ds + e^{(\beta_i + \lambda_{2i})t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(s)\, ds \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{d}{dt}\{e^{\beta_i t}[y_{1i}(t) - \underline{\phi}_i(t)]\} \\ &= (\beta_i + \lambda_{1i})\, e^{(\beta_i + \lambda_{1i})t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(s)\, ds \\ &\quad + (\beta_i + \lambda_{2i})\, e^{(\beta_i + \lambda_{2i})t} \int_{t}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(s)\, ds \\ &\quad + e^{(\beta_i + \lambda_{1i})t} \frac{e^{-\lambda_{1i} t}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(t) - e^{(\beta_i + \lambda_{2i})t} \frac{e^{-\lambda_{2i} t}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(t) \\ &= (\beta_i + \lambda_{1i})\, e^{(\beta_i + \lambda_{1i})t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(s)\, ds \\ &\quad + (\beta_i + \lambda_{2i})\, e^{(\beta_i + \lambda_{2i})t} \int_{i}^{+\infty} \frac{e^{-\lambda_{2i} s}}{d_i(\lambda_{2i} - \lambda_{1i})} h_i(s)\, ds \\ &\geqslant 0, \qquad t \in \mathbb{R}, \quad i = 1, \ldots, n \end{aligned}$$

since $\beta_i + \lambda_{2i} > 0$, $\beta_i + \lambda_{1i} > 0$ (by $c > 1 - \min\{\beta_i d_i;\ i = 1,..., n\}$) and $h_i(t) \geqslant 0$, $t \in \mathbb{R}$, $i = 1,..., n$. This completes the proof.

Combining Lemma 4.2 and Lemma 4.3, we know that $y_1 \in \Gamma^*$ and $e^{\beta t}[y_1(t) - \underline{\phi}(t)]$ is nondecreasing. Moreover, y_1 is also an upper solution of (2.3) by (4.1), Lemma 4.1, and Lemma 4.2. Thus we can repeat the above procedure. In general, we consider the following iteration scheme:

$$\begin{cases} cy'_m(t) = Dy''_m(t) - \beta y_m(t) + H(y_{m-1})(t), & m = 1, 2,... \\ y_0(t) = \bar{\phi}(t), & t \in \mathbb{R} \end{cases} \tag{4.7}$$

Solving (4.7) inductively, we obtain a sequence of vector functions $\{y_m(t)\}_{m=1}^{\infty}$ with the following properties:

(P1) $y_m \in \Gamma^*$, $m = 1, 2,...$;

(P2) $0 \leqslant \underline{\phi}(t) \leqslant y_m(t) \leqslant y_{m-1}(t) \leqslant \bar{\phi}(t)$, $t \in \mathbb{R}$, $m = 1, 2,...$;

(P3) $e^{\beta t}[y_m(t) - \underline{\phi}(t)]$ is nondecreasing in $\mathbb{R}$.

From (P2), we know that $\lim_{m \to \infty} y_m(t) = y(t)$ for $t \in \mathbb{R}$ exists and $\underline{\phi}(t) \leqslant y(t) \leqslant \bar{\phi}(t)$ for $t \in \mathbb{R}$. It is natural to expect the following.

Proposition 4.4. *$y(t)$ is a solution of (2.3) and (2.11).*

The proof of this proposition is similar to that of Proposition 3.5 and is omitted.

Summarizing Lemma 4.2, Lemma 4.3, and Proposition 4.4, we have proved the following.

Theorem 4.5. *Assume that (A1) and (A2)* hold. Suppose that (2.3) has an upper solution $\bar{\phi}$ in Γ^* and a lower solution $\underline{\phi}$ (which is not necessarily in Γ^*) satisfying (H1)*–(H3)*. Then (2.3) and (2.11) with $c > 1 - \min\{\beta_i d_i;\ i = 1,..., n\}$ have a solution in Γ^*. That is, (2.1) has a traveling wave front with speed $c > 1 - \min\{\beta_i d_i; i = 1,..., n\}$.*

Remark 4.6. In Theorem 3.6 and Theorem 4.5, the assumption $f(\hat{u}) \neq 0$ for $(\mathbf{0}, K)$ in (A1) as well as hypothesis (H2) or (H2)* is used only in proving $\lim_{t \to \infty} x(t) = K$ [or $\lim_{t \to \infty} y(t) = K$]. Therefore, any replacement to ensure $\lim_{t \to \infty} x(t) = K$ [or $\lim_{t \to \infty} y(t) = K$] will keep these two theorems remaining valid. Thus, we actually have the following.

Theorem 4.5*. *Assume that (A2)* holds and that $f(\mathbf{0}) = f(K) = \mathbf{0}$ with $\mathbf{0} < K$. Suppose that (2.3) has an upper solution $\bar{\phi}$ in Γ^* and a lower solution $\underline{\phi}$ (which is not necessarily in Γ^*) satisfying*

(H1) $\mathbf{0} \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant K,\ t \in \mathbb{R}$;

(H2)' $\underline{\phi}(t) \not\equiv \mathbf{0}$ *in* $\mathbb{R}$, *and there is no other equilibrium of (2.1) in* $[\delta, K]$, *where* $\delta = (\delta_1, ..., \delta_n)^T$ *with* $\delta_i = \sup_{t \in \mathbb{R}} \underline{\phi}_i(t)$, $i = 1, ..., n$;

(H3) $e^{\beta t}[\bar{\phi}(t) - \underline{\phi}(t)]$ *is nondecreasing in* $\mathbb{R}$.

Then (2.3) and (2.11) with $c > 1 - \min\{\beta_i d_i; i = 1, ..., n\}$ *have a solution in* Γ^*. *That is, (2.1) has a traveling wave front with speed* $c > 1 - \min\{\beta_i d_i; i = 1, ..., n\}$.

Theorem 3.6*. *Assume that (A2) holds and that* $f(\mathbf{0}) = f(K) = \mathbf{0}$ *with* $\mathbf{0} < K$. *Suppose that (2.3) has an upper solution* $\bar{\phi}$ *in* Γ *and a lower solution* $\underline{\phi}$ *(which is not necessarily in* γ*) satisfying*

(H1) $\mathbf{0} \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant k,\ t \in \mathbb{R}$;

(H2)' $\underline{\phi}(t) \not\equiv \mathbf{0}$ *in* $\mathbb{R}$, *and there is no other equilibrium of (2.1) in* $[\delta, K]$, *where* $\delta = (\delta_1, ..., \delta_n)^T$ *with* $\delta_i = \sup_{t \in \mathbb{R}} \underline{\phi}_i(t)$, $i = 1, ..., n$.

Then (2.3) and (2.11) have a solution in Γ. *That is, (2.1) has a traveling wave front.*

5. APPLICATIONS

Theorem 3.6 and Theorem 4.5 as well as their modifications reduce the problem of establishing the existence of traveling wave fronts to the existence of a pair of lower-upper solutions for an asymptotic boundary value problem. In what follows, we show that this reduction greatly simplifies the existence problem of traveling wave fronts by considering two models arising from different fields.

5.1. The Fisher–KPP Equation with Delay

The most classic and the simplest case of a nonlinear reaction-diffusion equation that was first shown to have traveling wave fronts is the so called Fisher–KPP equation [Fisher (1937), Kolmogorov, Petrovskii, and Piskunov, 1937)]

$$\frac{\partial u}{\partial t} = ru(t, x)[1 - u(t, x)] + D\,\frac{\partial^2 u}{\partial x^2} \tag{5.1.1}$$

where r and D are positive parameters. It was first suggested by Fisher (1937) as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population. It is also a natural extension of the logistic growth population ODE model.

Equation (5.1.1) can be normalized into the simpler form,

$$\frac{\partial u}{\partial t} = u(t, x)[1 - u(t, x)] + \frac{\partial^2 u}{\partial x^2} \tag{5.1.2}$$

by rescaling $t^* = rt$, $x^* = \sqrt{(K/D)}\, x$ and omitting the asterisks for notational simplicity.

Equation (5.1.2) and its traveling wave solutions have been widely studied [see, e.g., Kolmogorov *et al.* (1937), Fife (1979), Britton (1986), Murray (1989), Gardner (1995), and references therein], not only because it has in itself such wide applicability but also because it is the prototype equation which admits traveling wave front solutions. It is also a convenient equation from which many techniques can be developed for analyzing single-species models with spatial dispersal.

If we incorporate time delay into (5.1.2) as was done by many researchers for the corresponding logistic ODE model, we arrive at

$$\frac{\partial u}{\partial t} = u(t, x)[1 - u(t - \tau, x)] + \frac{\partial^2 u}{\partial x^2} \tag{5.1.3}$$

The corresponding ODE model, also called the Hutchinson equation, has been extensively studied in the literature [see Hutchinson (1948), Kuang (1933), So and Yu (1995), Sugie (1992), Wright (1955), and references therein].

Another way to incorporate the time delay is

$$\frac{\partial u}{\partial t} = u(t - \tau, x)[1 - u(t, x)] + \frac{\partial^2 u}{\partial x^2} \tag{5.1.4}$$

This equation was derived by K. Kobayshi (1977) from a branching process.

The existence of traveling wave fronts of (5.1.4) can be obtained by using the general theory of Schaaf (1987) and the monotone iteration technique developed by Zou and Wu (1997); due to the monotonicity of the reaction term with respect to the delayed argument. So in the sequel we concentrate on (5.1.3), to which the aforementioned methods fail to apply due to the nonmonotonicity of the nonlinear term of (5.1.3) with respect to the delayed argument.

The traveling wave equation corresponding to (2.3) in this case is

$$cx'(t) = x''(t) + x(t)[1 - x(t - c\tau)] \tag{5.1.5}$$

and the corresponding asymptotic boundary condition is

$$\lim_{t \to -\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t \to \infty} x(t) = 1 \tag{5.1.6}$$

In order to apply Theorem 4.5, we first note that (A1) is satisfied with $K=1$ and $f(\phi)=\phi(0)[1-\phi(0)(-\tau)]$. Next we show that $f_c(\phi)=\phi(0)[1-\phi(-c\tau)]$ satisfies (A2)*, provided that τ is sufficiently small.

Lemma 5.1.1. *If τ is sufficiently small, then $f_c(\phi)=\phi(0)[1-\phi(-c\tau)]$ satisfies (A2)*.*

Proof. Let ϕ, ψ be in $C([-c\tau;0];\mathbb{R})$ with $0\leqslant\psi(s)\leqslant\phi(s)\leqslant 1$, and $e^{\beta s}[\phi(s)-\psi(s)]$ nondecreasing for $s\in[-c\tau,0]$. Then

$$\begin{aligned}
f(\phi)-f(\psi) &= [\phi(0)-\psi(0)]-[\phi(0)\,\phi(-c\tau)-\psi(0)\,\psi(-c\tau)]\\
&= [\phi(0)-\psi(0)]-\phi(-c\tau)[\phi(0)-\psi(0)]\\
&\quad -\psi(0)[\phi(-c\tau)-\psi(-c\tau)]\\
&> [\phi(0)-\psi(0)]-[\phi(0)-\psi(0)]-e^{c\tau\beta}[\phi(0)-\psi(0)]\\
&= -e^{c\tau\beta}[\phi(0)-\psi(0)]
\end{aligned}$$

and hence

$$f(\phi)-f(\psi)+\beta[\phi(0)-\psi(0)]\geqslant(\beta-e^{\beta c\tau})[\phi(0)-\psi(0)]$$

Therefore, f satisfies (A2)* for $\beta>1$, provided that $\tau\geqslant 0$ is sufficiently small. This completes the proof. □

In the remainder of this section, we construct for (5.1.5) an upper solution and a lower solution satisfying the conditions in Theorem 4.5.

It is obvious that for $c>2$, $\Delta_{1c}(\lambda)=c\lambda-\lambda^2-1$ has exactly two real zeros,

$$0<\lambda_1:=\frac{c-\sqrt{c^2-4}}{2}<\lambda_2:=\frac{c+\sqrt{c^2-4}}{2} \tag{5.1.7}$$

and

$$\Delta_{1c}(\lambda)>0 \qquad \text{for} \quad \lambda\in(\lambda_1,\lambda_2)$$

Using λ_1, we first construct an upper solution of (5.1.5).

Proposition 5.1.2. *Let $c>2$ and $\lambda_1=(c-\sqrt{c^2-4})/2$. Then, for any $\alpha>0$,*

$$\bar{\phi}_\alpha(t)=\frac{1}{1+\alpha e^{-\lambda_1 t}}$$

is an upper solution of (5.1.5), provided that $\tau\geqslant 0$ is sufficiently small.

Proof. Direct calculation shows that

$$
\begin{aligned}
&\bar{\phi}''_\alpha(t)+\bar{\phi}_\alpha(t)[1-\bar{\phi}_\alpha(t-c\tau)]-c\bar{\phi}'_\alpha(t)\\
&\quad=\frac{\alpha e^{-\lambda_1 t}}{(1+\alpha e^{-\lambda_1 t})^3\,(1+\alpha e^{\lambda_1 c\tau}e^{-\lambda_1 t})}\,[\alpha^2 e^{\lambda_1 c\tau}e^{-\lambda_1 t}(\lambda_1^2+1-c\lambda_1)\\
&\qquad+\alpha e^{-\lambda_1 t}(\lambda_1^2-c\lambda_1+e^{\lambda_1 c\tau}(2-\lambda_1^2-c\lambda_1))+(e^{\lambda_1 c\tau}-\lambda_1^2-c\lambda_1)]
\end{aligned}
\tag{5.1.8}
$$

Employing $\lambda_1^2=c\lambda-1$, we can rewrite (5.1.8)

$$
\begin{aligned}
&\bar{\phi}''_\alpha(t)+\bar{\phi}_\alpha(t)[1-\bar{\phi}_\alpha(t-c\tau)]-c\bar{\phi}'_\alpha(t)\\
&\quad=\frac{\alpha e^{-\lambda_1 t}[-\alpha e^{-\lambda_1 t}(1+e^{\lambda_1 c\tau}(2c\lambda-1))-(2\lambda_1 c-1-e^{c\lambda_1\tau})]}{(1+\alpha e^{-\lambda_1 t})^3\,(1+\alpha e^{\lambda_1 c\tau}e^{-\lambda_1 t})}
\end{aligned}
\tag{5.1.9}
$$

Note that as a function of $c\in(2,\infty)$, $c\lambda_1$ is decreasing since

$$
\frac{d}{dc}(c\lambda_1)=\frac{-2(c-\sqrt{c^2-4})}{\sqrt{c^2-4}\,(c+\sqrt{c^2-4})}<0
$$

Furthermore, $\lim_{c\to 2}(c\lambda_1)=2$ and $\lim_{c\to\infty}(c\lambda_1)=1$. Thus, for any $c>2$, $c\lambda_1\in(1,2)$. On the other hand, for any $c>2$, we have that

$$
\begin{aligned}
[1+e^{c\lambda_1\tau}(2c\lambda_1-1)]_{\tau=0}&=2(c\lambda_1-1)>0\\
[2c\lambda_1-1-e^{c\lambda_1\tau}]_{\tau=0}&=2(c\lambda_1-1)>0
\end{aligned}
$$

Therefore, for any $c>2$, there exists $\tau_0(c)>0$ such that for $0\leqslant\tau\leqslant\tau_0(c)$, we have

$$
\begin{aligned}
1+e^{c\lambda_1\tau}(2c\lambda_1-1)&=2(c\lambda_1-1)>0\\
2c\lambda_1-1-e^{c\lambda_1\tau}&=2(c\lambda_1-1)>0
\end{aligned}
$$

These imply that

$$
\bar{\phi}''_\alpha(t)+\bar{\phi}_\alpha(t)[1-\bar{\phi}_\alpha(t-c\tau)]-c\bar{\phi}'_\alpha(t)<0,\qquad t\in\mathbb{R}
$$

This completes the proof.

Proposition 5.1.3. *Let $c>2$ and $0\leqslant\lambda_1\leqslant\lambda_2$ be defined by (5.1.7). Take $\varepsilon>0$ such that $\varepsilon<\lambda_1$ and $\lambda_1+\varepsilon<\lambda_2$. Ten, for sufficiently large $M>1$, $\underline{\phi}(t)=\max\{0,(1-Me^{\varepsilon t})\,e^{\lambda_1 t}\}$ is a lower solution of (5.15).*

Proof. Let $t_0 < 0$ be such that $Me^{\varepsilon t_0} = 1$.

(i) For $t > t_0$, $\underline{\phi}(t) = 0$ and $\underline{\phi}'(t) = \underline{\phi}''(t) = 0$. Therefore,

$$\underline{\phi}''(t) + \underline{\phi}(t)[1 - \underline{\phi}(t - c\tau)] - c\underline{\phi}'(t) = 0$$

(ii) For $t < t_0$,

$$\begin{aligned}
\underline{\phi}(t) &= (1 - Me^{\varepsilon t})\, e^{\lambda_1 t} \\
\underline{\phi}'(t) &= [\lambda_1 - M(\varepsilon + \lambda_1)\, e^{\varepsilon t}]\, e^{\lambda_1 t} \\
\underline{\phi}''(t) &= [\lambda_1 - M(\varepsilon + \lambda_1)^2\, e^{\varepsilon t}]\, e^{\lambda_1 t}
\end{aligned}$$

Hence

$$\begin{aligned}
&\underline{\phi}''(t) + \underline{\phi}(t)[1 - \underline{\phi}(t - c\tau)] - c\underline{\phi}'(t) \\
&\quad = e^{\lambda_1 t}[\lambda_1^2 - M(\varepsilon + \lambda_1)^2\, e^{\varepsilon t} + (1 - Me^{\varepsilon t}) - c(\lambda_1 - M(\varepsilon + \lambda_1)\, e^{\varepsilon t}) \\
&\qquad - (1 - Me^{\varepsilon t})(1 - Me^{\varepsilon(t - c\tau)})\, e^{\lambda_1 (t - c\tau)}] \\
&\quad \geqslant e^{\lambda_1 t}[\lambda_1^2 + 1 - c\lambda_1 - Me^{\varepsilon t}((\varepsilon + \lambda_1)^2 + 1 - c(\varepsilon + \lambda_1)) - e^{\lambda_1 (t - c\tau)}] \\
&\quad = e^{\lambda_1 t}[Me^{\varepsilon t} \Delta_{1c}(\varepsilon + \lambda_1) - e^{-\lambda_1 c\tau} e^{\lambda_1 t}] \\
&\quad \geqslant e^{\lambda_1 t}[Me^{\varepsilon t} \Delta_{1c}(\varepsilon + \lambda_1) - e^{-\lambda_1 c\tau} e^{\varepsilon t}] \\
&\quad = Me^{(\lambda_1 + \varepsilon)\, t} \left[\Delta_{1c}(\varepsilon + \lambda_1) - \frac{e^{-\lambda_1 c\tau}}{M} \right] \\
&\quad \geqslant 0
\end{aligned}$$

if $(e^{-\lambda_1 c\tau}/M) \leqslant \Delta_{1c}(\varepsilon + \lambda_1)$. This completes the proof.

Proposition 5.1.4. *Let $c > 2$ and λ_1, λ_2, ε, $\bar{\phi}_\alpha$, and $\underline{\phi}$ be as in Propositions 5.1.2 and 5.1.3. If $\beta \geqslant \lambda_1$, $0 < \alpha < \beta/2(\lambda_1 + \beta)$, and if $M > 1$ is sufficiently large such that*

$$\begin{cases} \sqrt{2} - 1 \leqslant \alpha M < M - 1 \\ M \geqslant \dfrac{e^{-c\lambda_1 \tau}}{\Delta_{1c}(\lambda_1 + \varepsilon)} \end{cases}$$

then

(i) $\bar{\phi}_\alpha \in \Gamma^*$;

(ii) $0 \leqslant \underline{\phi}(t) \leqslant \bar{\phi}_\alpha(t) \leqslant 1$, $t \in \mathbb{R}$;

(iii) $e^{\beta t}[\bar{\phi}_\alpha(t) - \underline{\phi}(t)]$ *is nondecreasing in* $t \in \mathbb{R}$.

The proof is a direct verification and is omitted here.

Combining Propositions 5.1.2–5.1.4 and applying Theorem 4.5, we obtain the following main result.

Theorem 5.1.5. *For any $c > 2$, there exists $\tau^*(c) > 0$ such that if $\tau \leqslant \tau^*(c)$, (5.1.3) has a traveling wave front solution with wave speed c.*

5.2. The Belousov–Zhabotinskii Reaction Model

The well-known Belousov–Zhabotinskii reaction, fiat discovered by Belousov (1959), is described by the system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)[1 - u(t,x) - rv(t,x)] \\[2ex] \dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial^2 v(t,x)}{\partial x^2} - bu(t,x)\, v(t,x) \end{cases} \tag{5.2.1}$$

where $r > 0$ and $b > 0$ are constants, and u and v correspond, respectively to the Bromic acid and bromide ion concentrations. This system was also derived and studied by Murray (1974, 1976) and can be regarded as a model for many other more complex biochemical and biological processes. Such processes are characterized (in the planar case) by the presence of circular waves that propagate with some constant speed c [see, e.g., Zaikin and Zhabotinskii (1970), Zhabotinskii (1974)].

There have been a number of research papers dealing with the existence of traveling wave front solutions of (5.2.1), where the waves move from a region of higher bromous acid concentration to one of lower bromous acid concentration as it reduces the level of the bromide ion [see, e.g., Ye and Wang (1987), Kanel (1990), Kapel (1991)]. In other words, with (5.2.1), the following boundary conditions have been proposed

$$\begin{cases} u(-\infty, t) = 0, \qquad v(-\infty, t) = 1 \\ u(+\infty, t) = 1, \qquad v(+\infty, t) = 0 \end{cases} \tag{5.2.2}$$

while looking for traveling front solutions of the form

$$\begin{cases} u(t,x) = \phi_1(s) \\ v(t,x) = \phi_2(s) \\ \qquad s = x + ct \end{cases} \tag{5.2.3}$$

where $c > 0$ is the wave speed.

There have been various arguments that time delay should be taken into consideration in biological models, especially in those population models, and much has been done in this aspect. As for chemical reaction models, little attention has been paid to the effect of time delay on the models. Recently, Zimmermann *et al.* (1984) derived a mathematical model for an illuminated thermochemical system where time delay was incorporated into the external feedback and found (theoretically and experimentally) that periodic attractors, which do not exist in the original system in the absence of delay, can be predicted to exist for longer delays. This reveals that even in chemical reaction models, time delay may play a role in the dynamics.

Due to the chemical and biological origin of (5.2.1) and the foregoing evidence, we incorporate a discrete delay $\tau \geqslant 0$ into system (5.2.1) to get

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)[1 - u(t,x) - rv(t-\tau,x)] \\ \dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial^2 v(t,x)}{\partial x^2} - bu(t,x)\, v(t,x) \end{cases} \tag{5.2.4}$$

Substituting (5.2.3) into (5.2.4) and (5.2.2), we get the equation for the profile functions ϕ_1 and ϕ_2

$$\begin{cases} c\phi_1'(t) = \phi_1''(t) + \phi_1(t)[1 - \phi_1(t) - r\phi_2(t - c\tau)] \\ c\phi_2'(t) = \phi_2''(t) - b\phi_1(t)\, \phi_2(t) \end{cases} \tag{5.2.5}$$

and

$$\begin{cases} \lim\limits_{t\to -\infty} \phi_1(t) = 0, \qquad \lim\limits_{t\to +\infty} \phi_1(t) = 1 \\ \lim\limits_{t\to -\infty} \phi_2(t) = 1, \qquad \lim\limits_{t\to +\infty} \phi_2(t) = 0 \end{cases} \tag{5.2.6}$$

Now, by making change of variables $\phi_2^* = 1 - \phi_2$ and still denoting it ϕ_2 for the convenience of notations, (5.2.5) and (5.2.6) become, respectively,

$$\begin{cases} c\phi_1'(t) = \phi_1''(t) + \phi_1(t)[s - \phi_1(t) + r\phi_2(t - c\tau)] \\ c\phi_2'(t) = \phi_2''(t) + b\phi_1(t)[1 - \phi_2(t)] \end{cases} \tag{5.2.7}$$

where $s = 1 - r$, and

$$\begin{cases} \lim\limits_{t\to -\infty} (\phi_1(t), \phi_2(t)) = (0,0) = \mathbf{0} \\ \lim\limits_{t\to -\infty} (\phi_1(t), \phi_2(t)) = (1,1) = \mathbf{1} \end{cases} \tag{5.2.8}$$

In the remainder of this section, we look for solutions to (5.2.7) and (5.2.8). In order to apply the theorems established in the previous sections, we first notice that $f(\phi)=(f_1(\phi), f_2(\phi))^T$ is defined by

$$f_1(\phi)=\phi_1(0)[s-\phi_1(0)+r\phi_2(-\tau)]$$
$$f_2(\phi)=b\phi_1(0)[1-\phi_2(0)]$$

We can verify that (A2) is satisfied by this f. In fact, for any $\phi=(\phi_1, \phi_2)$, $\psi=(\psi_1, \psi_2)\in C([-c\tau; 0]; \mathbb{R}^2)$ with $\mathbf{0}\leqslant\psi(\theta)\leqslant\phi(\theta)\leqslant\mathbf{1}$ for $\theta\in[-c\tau, 0]$, we have that

$$\begin{aligned} f_{c1}(\phi)-f_{c1}(\psi) &= s[\phi_1(0)-\psi_1(0)]-[\phi_1(0)+\psi_1(0)][\phi_1(0)-\psi_1(0)] \\ &\quad +r[\phi_1(0)\,\phi_2(-c\tau)-\psi_1(0)\,\psi_2(0)] \\ &\geqslant s[\phi_1(0)-\psi_1(0)]-2[\phi_1(0)-\psi_1(0)] \\ &= (s-2)[\phi_1(0)-\psi_1(0)] \end{aligned}$$

and

$$\begin{aligned} f_{c2}(\phi)-f_{c2}(\psi) &= b[\phi_1(0)-\psi_1(0)]-b[\phi_1(0)-\psi_1(0)]\,\phi_2(0) \\ &\quad -b\psi_1(0)[\phi_2(0)-\psi_2(0)] \\ &\geqslant b[\phi_1(0)-\psi_1(0)]-b[\phi_1(0)-\psi_1(0)]-b[\phi_2(0)-\psi_2(0)] \\ &= -b[\phi_2(0)-\psi_2(0)] \end{aligned}$$

Thus

$$f_c(\phi)-f_c(\psi)+\beta[\phi(0)-\psi(0)]\geqslant 0$$

where $\beta=diag(\beta_1, \beta_2)$ with $\beta_1\geqslant 2-s$ and $\beta_2\geqslant b$. So (A2) holds for this f.

It is easily seen that $\mathbf{0}=(0, 0)^T$ and $\mathbf{1}=(1, 1)^T$ are equilibria of (5.2.4). However, there are infinitely many other equilibria $(0, \theta)^T$, $\theta\in(0, 1]$, between $\mathbf{0}$ and $\mathbf{1}$. So, it is necessary for us to consider applying Theorem 3.6*. To this end, we need to find a pair of lower-upper solutions of (5.2.7) satisfying the conditions in Theorem 3.6*.

Lemma 5.2.1. *Assume that $b\leqslant s=1-r$. For any $c\geqslant 2\sqrt{s}$, let*

$$\lambda_1=\frac{c-\sqrt{c^2-4s}}{2}, \qquad \lambda_2=\frac{c+\sqrt{c^2+4}}{2}, \qquad \lambda_3=\frac{c+\sqrt{c^2+4b}}{2}$$

and

$$\bar{\phi}_1(t) = \min\{e^{\lambda_1 t}, 1\}$$

$$\bar{\phi}_2(t) = \bar{\phi}_1(t + c\tau) = \min\{e^{\lambda_1(t+c\tau)}, 1\}$$

$$\underline{\phi}_1(t) = \min\{\varepsilon e^{\lambda_2 t}, \varepsilon\}$$

$$\underline{\phi}_2(t) = \min\{\varepsilon e^{\lambda_3(t+c\tau)}, \varepsilon\}$$

where $0 < \varepsilon < 1$. *Then,* $\bar{\phi}(t) = (\bar{\phi}_1(t), \bar{\phi}_2(t))^T \in \Gamma$ *and is an upper solution of (5.2.7), and* $\underline{\phi}(t) = (\underline{\phi}_1(t), \underline{\phi}_2(t))^T$ *is a lower solution of (5.2.7) satisfying*

$$\mathbf{0} \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant \mathbf{1}, \qquad t \in \mathbb{R} \tag{5.2.9}$$

Proof. $\bar{\phi} \in \Gamma$ and (5.2.9) are obvious by the definition of $\bar{\phi}_i$ and $\underline{\phi}_i$ $(i = 1, 2)$ and by the fact that $\lambda_2 > \lambda_1 > 0$, $\lambda_3 > \lambda_1 > 0$ and $0 < \varepsilon < 1$. We divide the remaining part of the proof into two steps.

Step 1: $\bar{\phi}$ is an upper solution of (5.2.7).

(i) For $\bar{\phi}_1$: If $t > 0$, $\bar{\phi}_1(t) = 1$ and $\bar{\phi}_2(t - c\tau) = \bar{\phi}_1(t) = 1$. Hence,

$$c\bar{\phi}_1'(t) - \bar{\phi}_1''(t) - \bar{\phi}_1(t)[s - \bar{\phi}_1(t) + r\bar{\phi}_2(t - c\tau)] = 0$$

If $t < 0$, $\bar{\phi}(t) = e^{\lambda_1 t}$. Thus

$$\begin{aligned} & c\bar{\phi}_1'(t) - \bar{\phi}_1''(t) - \bar{\phi}_1(t)[s - \bar{\phi}_1(t) + r\bar{\phi}_2(t - c\tau)] \\ &= c\bar{\phi}_1'(t) - \bar{\phi}_1''(t) - s\bar{\phi}_1(t)[1 - \bar{\phi}_1(t)] \\ &= e^{\lambda_1 t}[c\lambda_1 - \lambda_1^2 - s + se^{\lambda_1 t}] \\ &\geqslant e^{\lambda_1 t}[c\lambda_1 - \lambda_1^2 - s] = 0 \end{aligned}$$

So, we have established that

$$c\bar{\phi}_1'(t) - \bar{\phi}_1''(t) - \bar{\phi}_1(t)[s - \bar{\phi}_1(t) + r\bar{\phi}_2(t - c\tau)] \geqslant 0, \qquad \text{a.e. in } \mathbb{R}$$

(ii) For $\bar{\phi}_2$: If $t > -c\tau$, $\bar{\phi}_2(t) = 1$. Hence,

$$c\bar{\phi}_2'(t) - \bar{\phi}_2''(t) - b\bar{\phi}_1(t)[1 - \bar{\phi}_2(t)] = 0$$

If $t < -c\tau$, then $\bar{\phi}_2(t) = e^{\lambda_1(t+c\tau)}$ and $\bar{\phi}_1(t) = e^{\lambda_1 t}$. Thus

$$\begin{aligned}
&c\bar{\phi}_2'(t) - \bar{\phi}_2''(t) - b\bar{\phi}_1(t)[1 - \bar{\phi}_2(t)] \\
&= e^{\lambda_1(t+c\tau)}[c\lambda_1 - \lambda_1^2 - be^{-\lambda_1 c\tau} + be^{\lambda_1 t}] \\
&\geqslant e^{\lambda_1(t+c\tau)}[s - be^{-\lambda_1 c\tau} + be^{\lambda_1 t}] \\
&\geqslant e^{b\lambda_1(t+c\tau)}[s - b] \geqslant 0
\end{aligned}$$

Therefore,

$$c\bar{\phi}_2'(t) - \bar{\phi}_2''(t) - b\bar{\phi}_1(t)[1 - \bar{\phi}_2(t)] \geqslant 0, \qquad \text{a.e. in } \mathbb{R}$$

Combining (i) and (ii), we conclude that $\bar{\phi}$ is an upper solution of (5.2.7).

Step 2: $\underline{\phi}$ is a lower solution of (5.2.7).

(i) For $\underline{\phi}_1$: If $t > 0$, $\underline{\phi}_1(t) = \varepsilon$ and $\underline{\phi}_2(t)(t - c\tau) = \varepsilon$, and hence

$$\begin{aligned}
&c\underline{\phi}_1'(t) - \underline{\phi}_1''(t) - \underline{\phi}_1(t)[s - \underline{\phi}_1(t) + r\underline{\phi}_2(t - c\tau)] \\
&= -\varepsilon[s - \varepsilon + r\varepsilon] \\
&= -\varepsilon s[1 - \varepsilon] < 0
\end{aligned}$$

If $t < 0$, $\underline{\phi}_1(t) = \varepsilon e^{\lambda_2 t}$ and thus

$$\begin{aligned}
&c\underline{\phi}_1'(t) - \underline{\phi}_1''(t) - \underline{\phi}_1(t)[s - \underline{\phi}_1(t) + r\underline{\phi}_2(t - c\tau)] \\
&\leqslant c\underline{\phi}_1'(t) - \underline{\phi}_1''(t) + \underline{\phi}_1^2(t) \\
&\leqslant c\underline{\phi}_1'(t) - \underline{\phi}_1''(t) + \underline{\phi}_1(t) \\
&= e^{\lambda_2 t}[c\lambda_2 - \lambda_2^2 + 1] = 0
\end{aligned}$$

Therefore,

$$c\underline{\phi}_1'(t) - \underline{\phi}_1''(t) - \underline{\phi}_1(t)[s - \underline{\phi}_1(t) + r\underline{\phi}_2(t - c\tau)] \leqslant 0, \qquad \text{a.e. in } \mathbb{R}$$

(ii) For $\underline{\phi}_2$: If $t > -c\tau$, $\underline{\phi}_2(t) = \varepsilon$, and hence

$$c\underline{\phi}_2'(t) - \underline{\phi}_2''(t) - b\underline{\phi}_1(t)[1 - \underline{\phi}_2(t)] = -b\underline{\phi}_1(t)[1 - \varepsilon] \leqslant 0$$

When $t \leqslant -c\tau$, $\underline{\phi}_2(t) = \varepsilon e^{\lambda_3(t+c\tau)}$, and thus

$$\begin{aligned} c\underline{\phi}_2'(t) - \underline{\phi}_2''(t) - b\underline{\phi}_1(t)[1-\underline{\phi}_2(t)] &\leqslant c\underline{\phi}_2'(t) - \underline{\phi}_2''(t) + b\underline{\phi}_1(t)\,\underline{\phi}_2(t) \\ &\leqslant c\underline{\phi}_2'(t) - \underline{\phi}_2''(t) + b\underline{\phi}_2(t) \\ &= e^{\lambda_3(t+c\tau)}[c\lambda_3 - \lambda_3^2 + b] = 0 \end{aligned}$$

Combining (i) and (ii), we conclude that $\underline{\phi}$ is a lower solution of (5.2.7). This completes the proof.

Finally, notice that $\delta_1 = \sup_{t\in\mathbb{R}} \underline{\phi}_1(t) = \varepsilon$, $\delta_2 = \sup_{t\in\mathbb{R}} \underline{\phi}_2(t) = \varepsilon$, and there is no other equilibrium of (5.2.7) in $[\delta, \mathbf{1}]$ where $\delta = (\varepsilon, \varepsilon)^T$ and $\mathbf{1} = (1, 1)^T$. Therefore, all the conditions in Theorem 3.6* are satisfied. This leads to the following

Theorem 5.2.2. *Assume that $0 < b \leqslant 1 - r$. Then for every $c \geqslant 2\sqrt{1-r}$ and $\tau > 0$, (5.2.4) has a traveling wave front solution with wave speed c.*

The following lemma is an analogue of Lemma 5.2.1 in the case when $1 - r < b$.

Lemma 5.2.3. *Assume that $s = 1 - r < b$. For every $c \geqslant 2\sqrt{b}$, let*

$$\lambda_0 = \frac{c - \sqrt{c^2 - 4b}}{2}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4}}{2}, \qquad \lambda_3 = \frac{c + \sqrt{c^2 + 4b}}{2}$$

and

$$\begin{aligned} \bar{\phi}_1(t) &= \min\{e^{\lambda_0 t}, 1\} \\ \bar{\phi}_2(t) &= \bar{\phi}_1(t + c\tau) = \min\{e^{\lambda_0(t+c\tau)}, 1\} \\ \underline{\phi}_1(t) &= \min\{\varepsilon e^{\lambda_2 t}, \varepsilon\} \\ \underline{\phi}_2(t) &= \min\{\varepsilon e^{\lambda_3(t+c\tau)}, \varepsilon\} \end{aligned}$$

where $0 < \varepsilon < 1$. Then $\bar{\phi}(t) = (\bar{\phi}_1(t), \bar{\phi}_2(t))^T \in \Gamma$ and is an upper solution of (5.2.7); and $\underline{\phi}(t) = (\underline{\phi}_1(t), \underline{\phi}_2(t))^T$ is a lower solution of (5.2.7) satisfying $\mathbf{0} \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant \mathbf{1}$ for $t \in \mathbb{R}$.

Proof. The proof is similar to that of Lemma 5.2.1 and is thus omitted here.

Applying Theorem 3.6* and Lemma 5.2.3, we have the following.

***Theorem* 5.2.4.** *Assume that* $1-r<b$. *Then, for every* $c \geqslant 2\sqrt{b}$ *and* $\tau>0$, *(5.2.4) has a traveling wave front with wave speed* c.

The following remarks compare our results with some existing ones.

Remark 5.2.5. Theorems 5.2.2 and 5.2.4 claim that the existence of traveling wave fronts for the Belousov–Zhabotinskii model (5.2.4) is independent of the delay. However, we should mention that the minimal wave velocity may be affected by the size of the delay.

Remark 5.2.6. When $\tau=0$, Troy (1980) investigated the existence of traveling wave fronts of $(5.2.4)_{\tau=0}$ by the shooting technique. The main result of Troy (1980) is as follows: For each $b \in (0, 1)$, there are values $c^* \in (0, 2]$ and $r^*>0$, such that $(5.2.4)_{\tau=0}$ with $c=c^*$ and $r=r^*$ has a traveling wave front. We can easily see that Theorem 5.2.2 implies this result and actually claims more. In fact, for each $b \in (0, 1)$, we can choose $r^*>0$ such that $0<b<1-r^*$. By Theorem 5.2.2, we know that for every $c \geqslant 2\sqrt{1-r^*}$, $(5.2.4)_{\tau=0}$ has a traveling wave front with wave speed c. In particular, for every $c \in [2\sqrt{1-r^*}, 2] \subset (0, 2]$, the same conclusion holds.

Remark 5.2.7. When $\tau=0$, Theorem 5.2.2 reduces to the main theorem of Ye and Wang (1987), which was obtained by first constructing a "nice" boundary value problem on a finite interval $[-a, a]$, and then passing to a limit by letting $a \to \infty$. Upper and lower solutions of that boundary value problem on $[-a, a]$ were also constructed, which would become invalid if $\tau \neq 0$.

Remark 5.2.8. Kanel (1990), and Kapel (1991) also studied the existence of traveling wave fronts of $(5.2.4)_{\tau=0}$. But their existence results were obtained under some further restrictions on b and r in addition to $r>0$ and $b>0$.

We conclude this paper by the following remark on the monotone iteration method and some related topics.

Remark 5.2.9. The monotone iteration technique has also been used before to construct wave fronts for integrodifferential equations (derivatives in time and integral in space) and integral equations (Valterra integrals in time and homogeneous integrals in space). See Diekmann (1978a–c) and Radcliffe and Rass (1983a, b). Although a reaction-diffusion system can be rewritten as an abstract integral equation using the fundamental solution of the heat operator, it does not seem to be possible to absorb the type of delay considered here into the integral kernel, and therefore, the resulting integral equation is not exactly of the same form as those considered in the just-mentioned papers. More importantly, it is not clear whether it is possible to obtain the required monotonicity structure in the integral equations. In particular, the use of nonstandard partial ordering in this paper seems to be novel in the context. On the other hand, in the integral equations framework it is possible to solve a couple of questions which are not addressed in this paper: the uniqueness of the wave front up to translation (Diekmann and Kaper, 1978; Radcliffe and Rass, 1983a, b) and the question which in the continuum of speeds is the one at which the solution of the original problem will finally travel. The second question is addressed in the theory of asymptotic speeds originally designed by Aronson and Weinberger (1975) for reaction-diffusion equations (*without delay*) and then adapted by Diekmann (1978b, 1979), Thieme (1979a, b), and Radcliffe and Rass (1986) to integral equations and by Weinberger (1981, 1982) and Lui (1988a, b) to abstract difference equations. Whether or not this theory can be adapted to reaction diffusion equations *with delay* remains open and will be discussed in a future paper.

ACKNOWLEDGMENTS

This research was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Alexander von Humboldt Foundation of Germany. The authors would like to thank the referee for her/his suggestions and comments on the paper and for bringing to our attention some related references.

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Printed in Belgium