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Stability and bifurcation of a reaction-diffusion-advection model with nonlinear boundary condition

Zhenzhen Li [a], Binxiang Dai [a,*], Xingfu Zou [b]

[a] *School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, 410083, PR China*
[b] *Department of Mathematics, University of Western Ontario, London, ON, N6A 5B7, Canada*

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Abstract

The dynamics of the reaction-diffusion-advection population models with linear boundary condition has been widely studied. This paper is devoted to the dynamics of a reaction-diffusion-advection population model with nonlinear boundary condition. Firstly, the stability of the trivial steady state is investigated by studying the corresponding eigenvalue problem. Secondly, the existence and stability of nontrivial steady states are proved by applying the Crandall-Rabinowitz bifurcation Theorem, the Lyapunov-Schmidt reduction method and perturbation method, in which bifurcation from simple eigenvalue and that from degenerate simple eigenvalue are both possible. The general results are applied to a parabolic equation with monostable nonlinear boundary condition, and to a parabolic equation with sublinear growth and superlinear boundary condition. Our theoretical results show that the nonlinear boundary condition can lead to the occurrence of various steady state bifurcations. Meanwhile, compared with the linear boundary condition, the nonlinear boundary condition can induce the multiplicity and growing-up property of positive steady-state solutions for the model with logistic interior growth. Finally, the numerical results show that the advection can change the bifurcation direction of some bifurcation, and affect the density distribution of the species.

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* Corresponding author.
E-mail addresses: zhzhli@csu.edu.cn (Z. Li), bxdai@csu.edu.cn (B. Dai), xzou@uwo.ca (X. Zou).

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1. Introduction

In this paper, we consider a general reaction-diffusion-advection population model with nonlinear boundary condition as follows:

$$\begin{cases} u_t = \nabla \cdot [d\nabla u - au\nabla P(x)] + F(x,u)u, & x \in \Omega, t > 0, \\ d\partial_{\vec{n}} u - au\partial_{\vec{n}} P = B(x,u)u, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \Omega. \end{cases} \tag{1.1}$$

Here u denotes the population density of species u at location x and time t; the habitat $\Omega \in \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $\vec{n}$ is the outward normal vector on the boundary $\partial\Omega$. The linear term $-d\nabla u + au\nabla P(x)$ with respect to u is called the flux of species u at location x and time t, where the parameter $d > 0$ is the random diffusion rate of species u, the non-constant function P may account for the abundance of a resource for the species u and hence the second term presents a resource driven advection effect with a being the advection rate. The reaction term $F(x,u)u$ stands for a general growth rate of species u. Our boundary condition of the form above means that when $B(x_0,u)u \geq 0$ for some $x_0 \in \partial\Omega$, the inflow rate of the population at the point x_0 to the region Ω is determined by $B(x_0,u)u$, while when $B(x_0,u)u < 0$ for some $x_0 \in \partial\Omega$, the individuals are taken outside the habitat at a rate $-B(x_0,u)u$ once they reach the boundary point x_0. Throughout this paper, we always assume that F, B and P satisfy the following conditions:

(H1) $F \in C^{1+\theta}(\overline{\Omega} \times \mathbb{R})$, $B \in C^{1+\theta}(\partial\Omega \times \mathbb{R})$ and $P \in C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$.

System (1.1) has important applications in several different biological scenarios. For instance, when $P(x) = m(x)$, $F(x,u) = m(x) - u$, $B(x,u) \equiv 0$, (1.1) reduces to

$$\begin{cases} u_t = \nabla \cdot [d\nabla u - au\nabla m(x)] + u[m(x) - u], & x \in \Omega, t > 0, \\ d\partial_{\vec{n}} u - au\partial_{\vec{n}} m = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $m(x)$ stands for the intrinsic growth rate of species u. Here the movement of species u is assumed to be a combination of random diffusion and directed motion upward along the gradient of the resource $m(x)$ reflected by describing the flux of species u at location x and time t as $-d\nabla u + au\nabla m$ (see [9]). It is known that persistence or extinction of species u in model (1.2) depends on the sign of the principal eigenvalue τ_1 of the linearized problem at $u = 0$ [12]:

$$\begin{cases} \nabla \cdot [d\nabla \psi - a\psi\nabla m(x)] + m(x)\psi + \tau\psi = 0, & x \in \Omega, \\ d\partial_{\vec{n}}\psi - a\psi\partial_{\vec{n}} m = 0, & x \in \partial\Omega. \end{cases}$$

If $\tau_1 > 0$, then $u = 0$ is stable and the population become extinct in the long run; if $\tau_1 < 0$, then $u = 0$ is unstable and the population grows exponentially near $u = 0$ at the rate of $e^{-\tau_1 t}$. Belgacem and Cosner [9] and subsequent work [19] have also shown that advection along resource gradients is beneficial to the persistence of a single species in convex habitats, while it is not necessary for non-convex habitats. One can refer to [14,15,17] for two competition species model with this type of advection term.

In a wide variety of environments, individuals are influenced by a constantly unidirectional flow (advection) which drives them out of the system, resulting in population decline. One of the most striking example is the aquatic organisms living in streams and rivers, where their dispersals are affected by the downstream water flow (see [30,33,34] and their references). If we consider a segment of river described by the interval $(0, L)$ and let $P(x) = x$, $F(x, u) = r(x) - u$, $\Omega = (0, L)$, $B(0, u) \equiv 0$ and $B(L, u) = -ba$, where b is positive constant, then (1.1) becomes

$$\begin{cases} u_t = du_{xx} - au_x + u[r(x) - u], & x \in (0, L), t > 0, \\ du_x(0, t) - au(0, t) = 0, & t > 0, \\ du_x(L, t) - au(L, t) = -bau(L, t), & t > 0, \\ u(x, 0) = u_0(x), & x \in (0, L). \end{cases} \tag{1.3}$$

Thus, Eq. (1.3) describes the population dynamics of a single species in such a river segment. Here at the upstream end $x = 0$, the no-flux boundary condition is assumed, meaning that no individuals will pass through this boundary. While at the downstream end $x = L$, the boundary condition contains a parameter $b \geq 0$ that measures the loss rate of individuals at the boundary relative to the flow rate (see [37] for a detailed derivation). Likewise, for model (1.3), the dynamics of species u is determined by the sign of the principal eigenvalue ν_1 of the linearized problem at $u = 0$ [12]:

$$\begin{cases} d\psi_{xx} - a\psi_x + r(x)\psi + \nu\psi = 0, & x \in \Omega, \\ d\psi_x(0) - a\psi(0) = 0, \\ d\psi_x(L) - a\psi(L) = -ba\psi(L). \end{cases}$$

Moreover, in the case of homogeneous environment, i.e. $r(x) \equiv const$, one can see from [42] that there always exists a critical advection rate such that the species can persist if and only if its advection rate is less than the critical rate. Recently, the dynamics of two species model with this type of advection term has been systematically studied by several authors, see, e.g., [41–43,66,67].

Note that in both (1.2) and (1.3), the reaction terms are logistic growth; and moreover, the functions prescribing the fluxes on the boundary are assumed linear in u (i.e., $B(x, u)$ is independent of u). In such a case, either there is no positive steady state and $u = 0$ is globally stable among nonnegative solutions, or there is a unique positive steady state which is globally stable. However, it was pointed out in [18] that any coefficient in a reaction-advection-diffusion model or its boundary conditions could depend on population density, leading to model of the form (1.1). When the function $B(x, u)$ is density dependent, properties of solutions may be dramatically different and bifurcation theory is a useful tool in this case. The current paper is devoted to study the more general model (1.1) in high spatial dimensions, where the interior reaction function $F(x, u)u$ and boundary reaction function $B(x, u)u$ are both nonlinear with respect to u.

1.1. Motivation and related work

Recently, Liu and Shi [38] studied the following scalar reaction-diffusion equation with nonlinear boundary condition (which can be viewed as a special non-advective version of the above model (1.1), i.e., $d = 1, a = 0$, $F(x, u) = \lambda k(x) f(u)$, $B(x, u) = \lambda r(x)b(u)$)

$$\begin{cases} u_t = \Delta u + \lambda k(x) f(u)u, & x \in \Omega, t > 0, \\ \partial_{\vec{n}} u = \lambda r(x) b(u)u, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \Omega, \end{cases} \tag{1.4}$$

where all parameters, biologically, can be understood in the same way as that in model (1.1), and the nonlinear boundary condition has the same meaning as that in model (1.1). Explored in [38] is the bifurcation of nontrivial steady state solutions of model (1.4) by applying some abstract local bifurcation theory in [20,21,39,40] as well as the global bifurcation theory in [49,54].

There are several aspects of investigations on reaction-diffusion models with nonlinear boundary conditions. The well-posedness and asymptotical behavior of solution are considered in [6,7,50–52]; the blow-up profiles of solutions are studied in [29,36,62]; and the boundary layer solutions are constructed in [8,11,22,23]. Existence, uniqueness and stability of steady state solutions for some special problems are also studied by using bifurcation method and other related methods [13,16,45,46,56–58,60,61]. Confined to (1.4), for example, when $f(u) = 0$ and the nonlinear term $b(u)u$ is logistic type, it is considered in [31,45,47]; when $f(u)u$ is logistic type and $b(u)u$ is superlinear type, it is considered in [13,16,57,58,60,61]. In these works, existence of positive branches of trivial solutions and their asymptotic behavior and stability under nonlinear boundary conditions are studied.

Motivated by the work of [38], a natural and fundamental question in this research direction would then be asked: can we establish a general bifurcation result for reaction-diffusion-advection model with nonlinear boundary condition, parallel as that for the non-advective model (1.4)? Having this question in mind, we then turn to discuss the dynamics of model (1.1).

1.2. Variable transformation and organization of the paper

For the convenience of analysis, we first make a variable transformation. Letting $\tilde{u} = e^{(-a/d)P(x)}u, t = \tilde{t}/d$, denoting $\lambda = 1/d, \alpha = a/d$, and dropping the tilde sign, model (1.1) is transformed to:

$$\begin{cases} u_t = e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla u \right] + \lambda f(x,u)u, & x \in \Omega, t > 0, \\ \partial_{\vec{n}} u = \lambda \beta(x,u)u, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \Omega, \end{cases} \tag{1.5}$$

where $f(x,u) = F(x, e^{\alpha P(x)}u)$ and $\beta(x,u) = B(x, e^{\alpha P(x)}u)$. The steady state solutions of (1.5) satisfy

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla u \right] = \lambda e^{\alpha P(x)} f(x,u)u, & x \in \Omega, \\ \partial_{\vec{n}} u = \lambda \beta(x,u)u, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

It is easy to see that (1.5) has a trivial steady state $u = 0$ for all $\lambda > 0$. That is, (1.6) has a line of trivial solutions

$$\Gamma_0 := \{(\lambda, 0) : \lambda > 0\} \tag{1.7}$$

in the $\lambda - u$ plane. For other steady states, note that when $\lambda = 0$ (i.e. $d \to \infty$ in (1.1)), any constant $u_1 \geq 0$ is a solution to (1.6). Hence (1.6) possesses the second line of trivial solutions:

$$\Gamma_{u_1} := \{(0, u_1) : u_1 > 0, u_1 \in \mathbb{R}\} \tag{1.8}$$

in the $\lambda - u$ plane.

In Section 2, we show some results for an eigenvalue problem with an indefinite weighted function, which play an important role in studying the stability and bifurcation of the steady state solutions of (1.5). Section 3 is devoted to the stability of the trivial steady state $u = 0$ of (1.5) by establishing the relation between the sign of the principal eigenvalue $\mu_1(\lambda, \hat{u})$ of (3.1) and the stability of steady states to (1.5). In Section 4, we derive three types of bifurcation phenomena and calculate the bifurcation direction and stability of the bifurcating positive steady states. These results indicate that non-trivial solutions of (1.6) can emerge from Γ_0 at some bifurcation point $(\lambda, u) = (\lambda_1, 0)$, or from Γ_{u_1} at some $(\lambda, u) = (0, u_1)$. In a special case, non-trivial solutions can also emerge from $(\lambda, u) = (0, 0)$, the intersection point of Γ_0 and Γ_{u_1}. Moreover, for illustration of our general results, we consider two examples of (1.5) in Sections 5 and 6 respectively and they are

(i) a parabolic equation with nonlinear boundary condition and monostable nonlinearity: $f(x, u) \equiv 0$ and $\beta(x, u)u = r(x)b(u)u$ with $b(u)$ satisfying $b(0) > 0$ and $b(1) = 0 > b'(1)$ (see Section 5); and
(ii) a parabolic equation with sublinear growth and superlinear boundary condition: $f(x, u)u = k(x)(u - u^p)$ and $\beta(x, u)u = r(x)u^q$ with $p, q > 1$ (see Section 6).

Finally, we summarize and discuss our results in Section 7.

Throughout the paper, we use the following notations. To consider the solutions of (1.6) in a functional setting, we define $X = W_l^2(\Omega)$, $Y = L^l(\Omega) \times W_l^{1-\frac{1}{l}}(\partial\Omega)$, with $l > N$. Denote the norm of the Banach space X by $\|\cdot\|$, and the duality pair of a Banach space X and its dual space X^* by $\langle \cdot, \cdot \rangle$. The notations $N(L)$ and $R(L)$ are used to denote the null space and the range space of linear operator L, respectively. Let $L[w]$ denote the image of w under the linear mapping L. For a multilinear operator L, we denote by $L[w_1, w_2, \cdots, w_k]$ the image of $(w_1, w_2, \cdots, w_k)$ under L, and when $w_1 = w_2 = \cdots = w_k$, we use $L[w_1]^k$ instead of $L[w_1, w_2, \cdots, w_k]$. For a nonlinear operator $\mathcal{F}$, we let $D_u\mathcal{F}$ denote the partial derivative of $\mathcal{F}$ with respect to u.

2. An eigenvalue problem with indefinite weighted function

In this section, we want to study the principal eigenvalue of an eigenvalue problem with indefinite weighted function. For this aim, we first comprehend the principal eigenvalue of the following linear eigenvalue problem for an eigenvalue $\mu(\lambda)$:

$$\begin{cases} -e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\phi\right] = \lambda g(x)\phi + \mu(\lambda)\phi, & x \in \Omega, \\ \partial_{\vec{n}}\phi = \lambda h(x)\phi, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $g \in C^\theta(\overline{\Omega})$ and $h \in C^{1+\theta}(\partial\Omega)$, $0 < \theta < 1$ are Hölder continuous functions in the closed domain $\overline{\Omega}$ and on the boundary $\partial\Omega$, respectively. Here, g and h may be both sign-changing, P is the same as in (1.1).

Now we state the following theorem concerning the existence of principal eigenvalue of (2.1) and its properties.

Theorem 2.1. *Assume that either* $\sup_\Omega g(x) > 0$ *or* $\sup_{\partial\Omega} h(x) > 0$. *Then we have the following conclusions:*

(i) For any $\lambda \in \mathbb{R}$, *problem* (2.1) *has a unique principal eigenvalue* $\mu_1(\lambda)$, *which is characterized variationally by*

$$\mu_1(\lambda) = \inf\left\{ \int_\Omega e^{\alpha P(x)}|\nabla\phi|^2 \mathrm{d}x - \lambda \int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x - \lambda \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S : \phi \in H^1(\Omega), \int_\Omega e^{\alpha P(x)}\phi^2 \mathrm{d}x = 1 \right\}. \tag{2.2}$$

Here $\mathrm{d}S$ *is the surface element of* $\partial\Omega$.

(ii) The mapping $\lambda \mapsto \mu_1(\lambda)$ *is concave and satisfies* $\mu_1(\lambda) \to -\infty$ *as* $\lambda \to \infty$.

(iii) If $\int_\Omega g\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\mathrm{d}S \le 0$, *then the principal eigenvalue* $\mu_1(\lambda)$ *has a unique local maximum (i.e. global maximum) with respect to* λ. *Moreover, the sign of the unique global maximum point is equal to that of* $-(\int_\Omega g\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\mathrm{d}S)$, *if it exists.*

When $\mu(\lambda) = 0$, (2.1) becomes an indefinite weighted eigenvalue problem as follows:

$$\begin{cases} -e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\phi\right] = \lambda g(x)\phi, & x \in \Omega, \\ \partial_{\vec{n}}\phi = \lambda h(x)\phi, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

We say that λ is a principal eigenvalue if (2.3) admits a positive solution (Notice that 0 is always a principal eigenvalue of (2.3)). The following theorem provides the existence and nonexistence of nonzero principal eigenvalue $\lambda_1(g, h)$ of (2.3).

Theorem 2.2.

(i) Assume that either $\sup_\Omega g(x) > 0$ *or* $\sup_{\partial\Omega} h(x) > 0$. *Then the problem* (2.3) *has a unique positive principal eigenvalue* $\lambda_1(g, h)$ *if and only if*

$$\int_\Omega e^{\alpha P(x)} g(x)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h(x)\mathrm{d}S < 0, \tag{2.4}$$

and it can be characterized by the following form

$$\lambda_1(g, h) = \inf\left\{ \frac{\int_\Omega e^{\alpha P(x)}|\nabla\phi|^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S} : \phi \in H^1(\Omega), \int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S > 0 \right\}. \tag{2.5}$$

(ii) *Assume that either* $\inf_\Omega g(x) < 0$ *or* $\inf_{\partial\Omega} h(x) < 0$. *Then the problem* (2.3) *has a unique negative principal eigenvalue* $\lambda_1(g, h)$ *if and only if*

$$\int_\Omega e^{\alpha P(x)} g(x)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h(x)\mathrm{d}S > 0.$$

(iii) *Assume that either* $g(x)$ *is sign-changing in* Ω *or* $h(x)$ *is sign-changing on* $\partial\Omega$. *If*

$$\int_\Omega e^{\alpha P(x)} g(x)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h(x)\mathrm{d}S = 0,$$

then 0 *is the unique principal eigenvalue of* (2.3).

2.1. Proof of Theorem 2.1

We will prove Theorem 2.1 by a variational argument similar to [55, Chapter 11]. Consider the minimizer of the functional

$$S_\lambda := \int_\Omega e^{\alpha P(x)} |\nabla\phi|^2 \mathrm{d}x - \lambda \int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x - \lambda \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S,$$

where $\phi \in H^1(\Omega)$ and $\int_\Omega e^{\alpha P(x)}\phi^2\mathrm{d}x = 1$. Recall the following result from [1, Lemma 1].

Proposition 2.3. *For any* $\varepsilon > 0$, *there exists a positive constant* $C(\varepsilon)$ *such that*

$$\int_{\partial\Omega} \phi^2 \mathrm{d}S \le \varepsilon \int_\Omega |\nabla\phi|^2 \mathrm{d}x + C(\varepsilon) \int_\Omega \phi^2 \mathrm{d}x, \quad \forall \phi \in H^1(\Omega).$$

Now we prove the following lemma.

Lemma 2.4. S_λ *has a lower bound.*

Proof. It follows from Proposition 2.3 that if $\int_\Omega e^{\alpha P(x)}\phi^2\mathrm{d}x = 1$, then it holds that

$$\left|\lambda \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S\right| \le \varepsilon|\lambda| \|e^{\alpha P(x)} h\|_{C(\partial\Omega)} \int_\Omega |\nabla\phi|^2 \mathrm{d}x + \frac{|\lambda| C(\varepsilon) \|e^{\alpha P(x)} h\|_{C(\partial\Omega)}}{e^{\alpha \min_{\overline{\Omega}} P(x)}}.$$

Thus we have

$$S_\lambda \ge \left(e^{\alpha \min_{\overline{\Omega}} P(x)} - \varepsilon|\lambda| \|e^{\alpha P(x)} h\|_{C(\partial\Omega)}\right) \int_\Omega |\nabla\phi|^2 \mathrm{d}x - |\lambda| \|g\|_{C(\overline{\Omega})} - \frac{|\lambda| C(\varepsilon) \|e^{\alpha P(x)} h\|_{C(\partial\Omega)}}{e^{\alpha \min_{\overline{\Omega}} P(x)}}.$$

This lemma is now proved if we take ε small enough such that $\varepsilon|\lambda| \|e^{\alpha P(x)} h\|_{C(\partial\Omega)} < e^{\alpha \min_{\overline{\Omega}} P(x)}$. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. (i) As in [55, Theorem 11.10], we can derive that the infimum of S_λ can be attained by some nonnegative and nonzero function $\phi_{1\lambda}$, which is of $C^{2+\theta}(\overline{\Omega})$ by elliptic regularity [25]. The maximum principle implies that $\phi_{1\lambda} > 0$ in Ω. If there exists $x_0 \in \partial\Omega$ such that $\phi_{1\lambda}(x_0) = 0$, then $\frac{\partial \phi_{1\lambda}}{\partial \vec{n}}\big|_{x=x_0} < 0$ by Hopf lemma, which contradicts the fact that $\frac{\partial \phi_{1\lambda}}{\partial \vec{n}}\big|_{x=x_0} = \lambda h(x_0)\phi_{1\lambda}(x_0) = 0$. Hence $\phi_{1\lambda} > 0$ in $\overline{\Omega}$. Here, the variational formula (2.2) can be proved by a similar manner as in [55, Theorem 11.4]. For the uniqueness of $\mu_1(\lambda)$, by way of contradiction, we suppose that (2.1) admits a principal eigenvalue $\mu(\neq \mu_1(\lambda))$, which is associated with an eigenfunction $\phi > 0$ on $\overline{\Omega}$. Then by integration by parts, we see that $\int_\Omega e^{\alpha P(x)}\phi\phi_1 \mathrm{d}x = 0$, which is a contradiction. Hence, part (i) is proved.

(ii) Consider the mapping

$$\lambda \longmapsto \int\limits_\Omega e^{\alpha P(x)}|\nabla\phi|^2 \mathrm{d}x - \lambda \int\limits_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x - \lambda \int\limits_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S$$

for a fixed $\phi \in H^1(\Omega)$. It is easy to see that this mapping is affine and then concave, so the infimum is also concave. Now we can choose a nontrivial function $\hat{\phi} \in C^1(\overline{\Omega})$ with $\int_\Omega e^{\alpha P(x)}\hat{\phi}^2 \mathrm{d}x = 1$, such that $\hat{\phi} > 0$ in $\{x \in \partial\Omega : h(x) > 0\}$ and the support of $\hat{\phi}$ is contained in a very thin tubular neighborhood of $\partial\Omega$ if $\sup\limits_{\partial\Omega} h > 0$, or that $\hat{\phi}$ has a compact support in $\{x \in \Omega : g(x) > 0\}$ if $\sup\limits_\Omega g > 0$. Based on above argument, we have

$$\mu_1(\lambda) \le \int\limits_\Omega e^{\alpha P(x)}|\nabla\hat{\phi}|^2 \mathrm{d}x - \lambda \int\limits_\Omega e^{\alpha P(x)} g\hat{\phi}^2 \mathrm{d}x - \lambda \int\limits_{\partial\Omega} e^{\alpha P(x)} h\hat{\phi}^2 \mathrm{d}S \to -\infty \text{ as } \lambda \to \infty.$$

This completes the proof of part (ii).

(iii) For part (iii), we see that

$$\begin{cases} -e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\phi_{1\lambda}\right] = \lambda g\phi_{1\lambda} + \mu_1(\lambda)\phi_{1\lambda}, & x \in \Omega, \\ \partial_{\vec{n}}\phi_{1\lambda} = \lambda h\phi_{1\lambda}, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

By differentiating (2.6) in λ, there holds that

$$\begin{cases} -e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\phi'_{1\lambda}\right] = g\phi_{1\lambda} + \lambda g\phi'_{1\lambda} + \mu'_1(\lambda)\phi_{1\lambda} + \mu_1(\lambda)\phi'_{1\lambda}, & x \in \Omega, \\ \partial_{\vec{n}}\phi'_{1\lambda} = h\phi_{1\lambda} + \lambda h\phi'_{1\lambda}, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

where the prime notation denotes the derivative with respective to λ. Multiplying (2.1) by $e^{\alpha P(x)}\phi'_{1\lambda}$ and (2.7) by $e^{\alpha P(x)}\phi_{1\lambda}$, subtracting the resulting equations and then integrating by parts over Ω, we derive

$$\mu'_1(\lambda) = -\frac{\int_\Omega e^{\alpha P(x)} g\phi_{1\lambda}^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi_{1\lambda}^2 \mathrm{d}S}{\int_\Omega e^{\alpha P(x)}\phi_{1\lambda}^2 \mathrm{d}x}. \tag{2.8}$$

Notice that $\phi_1(0)$ is a positive constant by (2.1). Then it follows from (2.8) that

$$\mu_1'(0) = -\frac{\int_\Omega e^{\alpha P(x)} g \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h \mathrm{d}S}{\int_\Omega e^{\alpha P(x)} \mathrm{d}x}. \tag{2.9}$$

Assume that $\int_\Omega e^{\alpha P(x)} g \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h \mathrm{d}S \le 0$. Then by conclusion (ii) and (2.9), we infer that $\mu_1(\lambda)$ has a critical point in $\lambda \ge 0$. In the following, we will show the uniqueness of the critical point for $\mu_1(\lambda)$. Let λ_0 be a critical point of $\mu_1(\lambda)$, that is, $\mu_1'(\lambda_0) = 0$ and $\int_\Omega e^{\alpha P(x)} g \phi_{1\lambda_0}^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h \phi_{1\lambda_0}^2 \mathrm{d}S = 0$. In this case, we normalize the eigenfunction $\phi_{1\lambda_0}$ as $\int_\Omega e^{\alpha P(x)} \phi_{1\lambda_0}^2 \mathrm{d}x = 1$. In view of the concavity of $\mu_1(\lambda)$, we only need to prove that $\mu_1(\lambda) < \mu_1(\lambda_0)$ if $\lambda \ne \lambda_0$. It is easy to see that $\mu_1(\lambda_0) = \int_\Omega e^{\alpha P(x)} |\nabla \phi_{1\lambda_0}|^2 \mathrm{d}x$. From the definition of $\mu_1(\lambda)$, there holds that for any $\lambda \ne \lambda_0$,

$$\begin{aligned} \mu_1(\lambda) &\le \int_\Omega e^{\alpha P(x)} |\nabla \phi_{1\lambda_0}|^2 \mathrm{d}x - \lambda \int_\Omega e^{\alpha P(x)} g \phi_{1\lambda_0}^2 \mathrm{d}x - \lambda \int_{\partial\Omega} e^{\alpha P(x)} h \phi_{1\lambda_0}^2 \mathrm{d}S \\ &= \int_\Omega e^{\alpha P(x)} |\nabla \phi_{1\lambda_0}|^2 \mathrm{d}x = \mu_1(\lambda_0). \end{aligned}$$

Suppose to the contrary that there is a $\lambda_1 \ne \lambda_0$ such that $\mu_1(\lambda_1) = \mu_1(\lambda_0)$, then the infimum of S_{λ_1} can be achieved by $\phi_{1\lambda_0}$, which means

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \phi_{1\lambda_0} \right] = \lambda_1 g \phi_{1\lambda_0} + \mu_1(\lambda_1) \phi_{1\lambda_0}, & x \in \Omega, \\ \partial_{\vec{n}} \phi_{1\lambda_0} = \lambda_1 h \phi_{1\lambda_0}, & x \in \partial\Omega. \end{cases}$$

We also notice that

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \phi_{1\lambda_0} \right] = \lambda_0 g \phi_{1\lambda_0} + \mu_1(\lambda_0) \phi_{1\lambda_0}, & x \in \Omega, \\ \partial_{\vec{n}} \phi_{1\lambda_0} = \lambda_0 h \phi_{1\lambda_0}, & x \in \partial\Omega. \end{cases}$$

By the assumption that $\mu_1(\lambda_0) = \mu_1(\lambda_1)$, we have

$$(\lambda_1 - \lambda_0) g \phi_{1\lambda_0} = 0 \ \text{ for } x \in \Omega, \quad \text{and} \ \ (\lambda_1 - \lambda_0) h \phi_{1\lambda_0} = 0 \ \text{ for } x \in \partial\Omega.$$

Then the positivity of $\phi_{1\lambda_0}$ implies that $g \equiv 0$ for $x \in \Omega$ and $h \equiv 0$ for $x \in \partial\Omega$, a contradiction. Hence we have proved the uniqueness of the local critical point. Finally, it follows from (2.9) that the sign of the unique global maximum point is the same as that of $-(\int_\Omega e^{\alpha P(x)} g \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h \mathrm{d}S)$. The proof is completed. □

2.2. *Proof of Theorem 2.2*

To prove Theorem 2.2, we need to verify that the infimum (2.5) is well-defined by a positive constant. Recall the following result from [59, Lemma 4.1].

Lemma 2.5. *Suppose that $\int_\Omega e^{\alpha P(x)} g(x) \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h(x) \mathrm{d}S < 0$. If $\phi \in H^1(\Omega)$ satisfies that $\int_\Omega \phi^2 \mathrm{d}x + \int_{\partial\Omega} \phi^2 \mathrm{d}S = 1$ and $\int_\Omega e^{\alpha P(x)} g \phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h \phi^2 \mathrm{d}S > 0$, then there is a constant c_0 such that $\int_\Omega |\nabla \phi|^2 \mathrm{d}x \ge c_0$.*

Suppose that $\phi \in H^1(\Omega)$ satisfies $\int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S > 0$ and let the function $\varphi = \delta\phi$ be with $\int_\Omega \varphi^2 \mathrm{d}x + \int_{\partial\Omega} \varphi^2 \mathrm{d}S = 1$ for some $\delta > 0$. Then there holds that

$$\begin{aligned}\frac{\int_\Omega e^{\alpha P(x)} |\nabla\phi|^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S} &\geq \frac{e^{\alpha \min_{\overline{\Omega}} P(x)} \int_\Omega |\nabla\phi|^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S} \\ &= \frac{e^{\alpha \min_{\overline{\Omega}} P(x)} \int_\Omega |\nabla\varphi|^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} g\varphi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\varphi^2 \mathrm{d}S} \\ &\geq \frac{c_0 e^{\alpha \min_{\overline{\Omega}} P(x)}}{\|e^{\alpha P(x)} g^+\|_{C(\overline{\Omega})} + \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)}} > 0, \end{aligned} \tag{2.10}$$

where $f^+ := \max\{f, 0\}$ for a continuous function f. This inequality shows that the infimum (2.5) is positive.

Proof of Theorem 2.2. (i) Let λ_* be a positive constant defined by the infimum in (2.5). Notice that $\mu_1(0) = 0$ from (2.1). It follows from Theorem 2.1 (ii) and (iii) that once $\int_\Omega e^{\alpha P(x)} g(x) \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h(x) \mathrm{d}S < 0$, then there exists a unique positive principal eigenvalue $\lambda_1(g, h)$ of (2.3), i.e., $\mu_1(\lambda_1(g, h)) = 0$, while (2.3) admits no principal eigenvalue for any $\lambda \neq \lambda_1(g, h)$. In view of this point, to obtain formula (2.5), we only need to prove that $\mu_1(\lambda_*) = 0$. For $\phi \in H^1(\Omega)$ satisfying $\int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S > 0$, by the definition of λ_*, we have

$$\int_\Omega e^{\alpha P(x)} |\nabla\phi|^2 \mathrm{d}x - \lambda_* \int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x - \lambda_* \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S \geq 0. \tag{2.11}$$

Since $\lambda_* > 0$, the inequality (2.11) also holds true when $\int_\Omega e^{\alpha P(x)} g\phi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi^2 \mathrm{d}S \leq 0$, which implies that $\mu_1(\lambda_*) \geq 0$.

On the other hand, by the definition of the infimum in (2.5), there is a sequence $\{\phi_n\} \subset H^1(\Omega)$ such that

$$\int_\Omega e^{\alpha P(x)} g\phi_n^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi_n^2 \mathrm{d}S > 0,$$

$$\int_\Omega \phi_n^2 \mathrm{d}x = 1, \tag{2.12}$$

$$\left(1 + \frac{1}{n}\right)\lambda_* \geq \frac{\int_\Omega e^{\alpha P(x)} |\nabla\phi_n|^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} g\phi_n^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h\phi_n^2 \mathrm{d}S}. \tag{2.13}$$

It follows from Proposition 2.3 that for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

$$\begin{aligned}\int_{\partial\Omega} e^{\alpha P(x)} h\phi_n^2 \mathrm{d}S &\leq \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)} \int_{\partial\Omega} \phi_n^2 \mathrm{d}S \\ &\leq \varepsilon \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)} \int_\Omega |\nabla\phi_n|^2 \mathrm{d}x + C(\varepsilon) \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)} \int_\Omega \phi_n^2 \mathrm{d}x. \end{aligned} \tag{2.14}$$

This together with (2.12) and (2.13) leads to that

$$\left(e^{\alpha \min_{\overline{\Omega}} P(x)} - 2\lambda_* \varepsilon \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)}\right) \int_\Omega |\nabla \phi_n|^2 \mathrm{d}x \le 2\lambda_* (\|e^{\alpha P(x)} g^+\|_{C(\overline{\Omega})} + C(\varepsilon)\|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)}),$$

where we have used that $1 + 1/n \le 2$. We can choose $\varepsilon > 0$ small enough such that

$$e^{\alpha \min_{\overline{\Omega}} P(x)} - 2\lambda_* \varepsilon \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)} > \frac{e^{\alpha \min_{\overline{\Omega}} P(x)}}{2},$$

and hence

$$\int_\Omega |\nabla \phi_n|^2 \mathrm{d}x \le \frac{4\lambda_*\left(\|e^{\alpha P(x)} g^+\|_{C(\overline{\Omega})} + C(\varepsilon)\|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)}\right)}{e^{\alpha \min_{\overline{\Omega}} P(x)}} < \infty.$$

This combined with (2.12) implies that ϕ_n is bounded in $H^1(\Omega)$, and also bounded in $L^2(\partial\Omega)$ by virtue of the continuous imbedding $H^1(\Omega) \subset L^2(\partial\Omega)$. Now we can derive from (2.13) that

$$\begin{aligned}
&\int_\Omega e^{\alpha P(x)} |\nabla \phi_n|^2 \mathrm{d}x - \lambda_* \int_\Omega e^{\alpha P(x)} g \phi_n^2 \mathrm{d}x - \lambda_* \int_{\partial\Omega} e^{\alpha P(x)} h \phi_n^2 \mathrm{d}S \\
&\le \frac{\lambda_*}{n} \left(\int_\Omega e^{\alpha P(x)} g \phi_n^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} h \phi_n^2 \mathrm{d}S \right) \\
&\le \frac{\lambda_*}{n} \left(\|e^{\alpha P(x)} g^+\|_{C(\overline{\Omega})} + \|e^{\alpha P(x)} h^+\|_{C(\partial\Omega)} \int_{\partial\Omega} \phi_n^2 \mathrm{d}S \right) \to 0, \quad n \to \infty,
\end{aligned}$$

which leads to that $\mu_1(\lambda_*) \le 0$ and hence we obtain $\mu_1(\lambda_*) = 0$. This proves part (i).

For part (ii), we denote $p(x) = -g(x)$ for $x \in \Omega$ and $q(x) = -h(x)$ for $x \in \partial\Omega$. Since either $\inf_\Omega g(x) < 0$ or $\inf_{\partial\Omega} h(x) < 0$, it follows from part (i) that $\lambda_1(p, q) > 0$. Clearly, $-\lambda_1(p, q)$ is the unique negative principal eigenvalue of (2.3). This proves part (ii). The conclusion of part (iii) can be inferred from parts (i) and (ii) directly. The proof for Theorem 2.2 is completed. □

3. Stability of trivial steady states

In this section, we will discuss the stability of the trivial solution $u = 0$ of (1.6). Let $\hat{u}$ be a nonnegative solution of (1.6). By linearizing system (1.5) at $\hat{u}$, we can obtain the linear eigenvalue problem

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi\right] - \lambda\left[f(x, \hat{u}) + \hat{u} f_u(x, \hat{u})\right]\psi = \mu(\lambda, \hat{u})\psi, & x \in \Omega, \\ \partial_{\vec{n}} \psi - \lambda\left[\beta(x, \hat{u}) + \hat{u}\beta_u(x, \hat{u})\right]\psi = \mu(\lambda, \hat{u})\psi, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

It follows from Theorem 2.2 in [3] that the eigenvalue problem (3.1) admits a unique principal eigenvalue, which is simple with a positive eigenfunction on $\overline{\Omega}$. Denote the principal eigenvalue and the corresponding eigenfunction of problem (3.1) by $\mu_1(\lambda, \hat{u})$ and $\psi_1(\lambda, \hat{u})$, respectively.

Definition 3.1. A function $\tilde{v} \in C^{2,1}(\overline{\Omega} \times (0, \infty))$ is called a upper-solution of (1.5) if $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t \geq e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \tilde{v} \right] + \lambda f(x, \tilde{v}) \tilde{v}, & x \in \Omega, t > 0, \\ \partial_{\vec{n}} \tilde{v} \geq \lambda \beta(x, \tilde{v}) \tilde{v}, & x \in \partial\Omega, t > 0, \\ \tilde{v}(x, 0) \geq u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{3.2}$$

A lower-solution of (1.5) is defined by reversing all the inequalities in (3.2).

Firstly, we show the relation between the sign of the principal eigenvalue $\mu_1(\lambda, \hat{u})$ of (3.1) and the stability of steady states to (1.5).

Proposition 3.2.

(i) Let $\hat{u}$ be a nonnegative steady state of (1.5) *and $\delta > 0$ be a constant. If $\mu_1(\lambda, \hat{u}) > 0$, then there exists a constant $\rho > 0$ such that* (1.5) *admits a unique global solution u satisfying*

$$|u(x, t) - \hat{u}(x)| \leq \rho e^{-\delta t} \psi_1(x), \quad \forall t > 0, x \in \overline{\Omega}$$

provided a nonnegative and not identically zero initial data $u_0 \in C^2(\overline{\Omega})$ with the condition

$$|u_0(x) - \hat{u}(x)| \leq \rho \psi_1(x), \quad \forall x \in \overline{\Omega},$$

that is, the steady state $\hat{u}$ is locally asymptotically stable. Here ψ_1 is the positive eigenfunction associated with the principal eigenvalue $\mu_1(\lambda, \hat{u})$ of (3.1)*.*

(ii) Let $\hat{u}$ be a nonnegative steady state of (1.5) *and $0 < \sigma < 1$ be a constant. Suppose that $\mu_1(\lambda, \hat{u}) < 0$. Then there exist constants $\hat{\rho}_0 = \hat{\rho}_0(\mu_1), \hat{\delta} = \hat{\delta}(\sigma, \mu_1) > 0$ such that if $0 < \rho \leq \hat{\rho}_0$, then any solution u of* (1.5) *satisfies*

$$u(x, t) \leq \hat{u}(x) - \rho(1 - \sigma e^{-\hat{\delta} t}) \psi_1(x), \quad \forall t > 0, x \in \overline{\Omega} \tag{3.3}$$

provided a nonnegative and not identically zero initial data $u_0 \in C^2(\overline{\Omega})$ with the condition

$$u_0(x) \leq \hat{u}(x) - \rho(1 - \sigma) \psi_1(x), \quad \forall x \in \overline{\Omega}.$$

Meanwhile, there exist constant $\tilde{\rho}_0 = \tilde{\rho}_0(\mu_1), \tilde{\delta} = \tilde{\delta}(\sigma, \mu_1) > 0$ such that if $0 < \rho \leq \tilde{\rho}_0$, then any solution u of (1.5) *satisfies*

$$u(x, t) \geq \hat{u}(x) + \rho(1 - \sigma e^{-\tilde{\delta} t}) \psi_1(x), \quad \forall t > 0, x \in \overline{\Omega} \tag{3.4}$$

provided a nonnegative and not identically zero initial data $u_0 \in C^2(\overline{\Omega})$ with the condition

$$u_0(x) \geq \hat{u}(x) + \rho(1 - \sigma) \psi_1(x), \quad \forall x \in \overline{\Omega}.$$

That is, the steady state $\hat{u}$ is unstable.

Proof. The proof is similar to that of [48, Theorem 5.3.3] with a minor modification. At first, we prove conclusion (i). Set $\tilde{v}(x,t) = \hat{u}(x) + \rho e^{-\delta t}\psi_1(x)$ with positive constants ρ, δ. As in the proof of [48, Theorem 5.3.3], we can infer that there exists a positive ρ_1 such that for any $\rho \in (0, \rho_1]$,

$$\tilde{v}_t - e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\tilde{v}\right] - \lambda f(x,\tilde{v})\tilde{v} \geq 0 \text{ for } t > 0 \text{ and } x \in \overline{\Omega}. \tag{3.5}$$

For the boundary condition, we see that

$$\begin{aligned}\partial_{\vec{n}}\tilde{v} - \lambda\beta(x,\tilde{v})\tilde{v} &= \lambda\beta(x,\hat{u})\hat{u} + \rho e^{-\delta t}\Big[\lambda\big[\beta(x,\hat{u}) + \hat{u}\beta_u(x,\hat{u})\big]\psi_1 + \mu_1(\lambda,\hat{u})\psi_1\Big] \\ &\quad - \lambda\beta\Big(x, \hat{u} + \rho e^{-\delta t}\psi_1\Big)\Big(\hat{u} + \rho e^{-\delta t}\psi_1\Big) \\ &= \rho e^{-\delta t}\psi_1\Big(\mu_1(\lambda,\hat{u}) - O(|\rho e^{-\delta t}\psi_1|)\Big)\end{aligned}$$

for $t > 0$ and $x \in \partial\Omega$. It follows from the above identity that there exists a positive constant ρ_2 such that for any $\rho \in (0, \rho_2]$,

$$\partial_{\vec{n}}\tilde{v} - \lambda\beta(x,\tilde{v})\tilde{v} \geq 0 \text{ for } t > 0 \text{ and } x \in \partial\Omega. \tag{3.6}$$

The two inequalities (3.5) and (3.6) together with the initial data $u_0 \leq \hat{u} + \rho\psi_1$ imply that $\tilde{v} = \hat{u} + \rho e^{-\delta t}\psi_1$ is an upper-solution of (1.5) if $0 < \rho \leq \min\{\rho_1, \rho_2\}$.

By the same argument as above, it can be inferred that $\hat{v}(x,t) = \hat{u}(x) - \rho e^{-\delta t}\psi_1$ is a lower-solution of (1.5) provided that $u_0 \geq \hat{u} - \rho\psi_1$, where $0 < \rho \leq \rho_3$ for some positive constant ρ_3. Now, conclusion (i) can be obtained from [48, Theorem 4.1.1].

In the following, we only show the former part of conclusion (ii), as the latter part can be checked by a similar manner. Set $\tilde{v}(x,t) = \hat{u}(x) - \rho(1 - \sigma e^{-\delta t})\psi_1(x)$ with constant $0 < \sigma < 1, \rho > 0, \delta > 0$. As in the proof of [48, Theorem 5.3.3], there exist constants $\rho_1, \delta_1 > 0$ such that if $\rho \in (0, \rho_1]$ and $\delta = \delta_1$, then it holds that

$$\tilde{v}_t - e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\tilde{v}\right] - \lambda f(x,\tilde{v})\tilde{v} \geq 0 \text{ for } t > 0 \text{ and } x \in \overline{\Omega}. \tag{3.7}$$

For the boundary condition, we have

$$\partial_{\vec{n}}\tilde{v} - \lambda\beta(x,\tilde{v})\tilde{v} = \rho(1 - \sigma e^{-\delta t})\psi_1\Big(O(|\rho(1 - \sigma e^{-\delta t})\psi_1|) - \mu_1(\lambda,\hat{u})\Big).$$

Then we can find the constants $\rho_2 > 0$ and $0 < c_1 < -\mu_1(\lambda,\hat{u})$ such that for $\rho \in (0, \rho_2]$,

$$\partial_{\vec{n}}\tilde{v} - \lambda\beta(x,\tilde{v})\tilde{v} \geq -(c_1 + \mu_1(\lambda,\hat{u}))\rho(1 - \sigma e^{-\delta t})\psi_1 \geq 0 \text{ for } t > 0 \text{ and } x \in \partial\Omega. \tag{3.8}$$

Again, the two inequalities (3.7) and (3.8) show that $\tilde{v} = \hat{u} - \rho(1 - \sigma e^{-\delta t})\psi_1$ is an upper-solution of (1.5) given that $u_0 \leq \hat{u} - \rho(1 - \sigma)\psi_1$, where $0 < \rho \leq \min\{\rho_1, \rho_2\}$ and $\delta = \delta_1$. Consequently, (3.3) follows from the comparison argument as developed in [48, Theorem 4.1.2]. The proof of this theorem is completed. □

To classify the main result of this section, we need to state the following lemma. Here we denote by ϕ_1 the normalized positive eigenfunction associated with principal eigenvalue λ_1 of (2.3) with g replaced by $f(x,\hat{u})+\hat{u}f_u(x,\hat{u})$ and h replaced by $\beta(x,\hat{u})+\hat{u}\beta_u(x,\hat{u})$. And we also define ψ_1 as the normalized positive eigenfunction corresponding to the principal eigenvalue $\mu_1(\lambda,\hat{u})$ of (3.1).

Lemma 3.3. *The principal eigenvalue $\mu_1(\lambda,\hat{u})$ of (3.1) satisfies the following two identities:*

$$\begin{aligned}
&\mu_1(\lambda,\hat{u})\left(\int_\Omega e^{\alpha P(x)}\phi_1^2\mathrm{d}x+\int_{\partial\Omega}e^{\alpha P(x)}\phi_1^2\mathrm{d}S\right)\\
&\quad=-\int_\Omega e^{\alpha P(x)}\psi_1^2\left|\nabla\left(\frac{\phi_1}{\psi_1}\right)\right|^2\mathrm{d}x+(\lambda_1-\lambda)\int_\Omega e^{\alpha P(x)}\phi_1^2\big(f(x,\hat{u})+f_u(x,\hat{u})\hat{u}\big)\mathrm{d}x\\
&\qquad+(\lambda_1-\lambda)\int_{\partial\Omega}e^{\alpha P(x)}\phi_1^2\big(\beta(x,\hat{u})+\beta_u(x,\hat{u})\hat{u}\big)\mathrm{d}S,
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&\mu_1(\lambda,\hat{u})\left(\int_\Omega e^{\alpha P(x)}\psi_1^2\mathrm{d}x+\int_{\partial\Omega}e^{\alpha P(x)}\psi_1^2\mathrm{d}S\right)\\
&\quad=\int_\Omega e^{\alpha P(x)}\phi_1^2\left|\nabla\left(\frac{\psi_1}{\phi_1}\right)\right|^2\mathrm{d}x+(\lambda_1-\lambda)\int_\Omega e^{\alpha P(x)}\psi_1^2\big(f(x,\hat{u})+f_u(x,\hat{u})\hat{u}\big)\mathrm{d}x\\
&\qquad+(\lambda_1-\lambda)\int_{\partial\Omega}e^{\alpha P(x)}\psi_1^2\big(\beta(x,\hat{u})+\beta_u(x,\hat{u})\hat{u}\big)\mathrm{d}S.
\end{aligned}\tag{3.10}$$

Proof. Notice the equality

$$\begin{aligned}
\nabla\cdot\left[e^{\alpha P(x)}\psi_1^2\nabla\left(\frac{\phi_1}{\psi_1}\right)\right]&=e^{\alpha P(x)}\psi_1(\nabla P\cdot\nabla\phi_1+\Delta\phi_1)-e^{\alpha P(x)}\phi_1(\nabla P\cdot\nabla\psi_1+\Delta\psi_1)\\
&=\psi_1\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]-\phi_1\nabla\cdot\left[e^{\alpha P(x)}\nabla\psi_1\right].
\end{aligned}$$

Multiplying the above equation by $\dfrac{\phi_1}{\psi_1}$ and making an integration over Ω, we see that

$$\begin{aligned}
&\int_{\partial\Omega}\frac{\phi_1}{\psi_1}e^{\alpha P(x)}(\psi_1\partial_{\vec{n}}\phi_1-\phi_1\partial_{\vec{n}}\psi_1)\mathrm{d}S-\int_\Omega e^{\alpha P(x)}\psi_1^2\left|\nabla\left(\frac{\phi_1}{\psi_1}\right)\right|^2\mathrm{d}x\\
&\quad=\int_\Omega\phi_1\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]\mathrm{d}x-\int_\Omega\frac{\phi_1^2}{\psi_1}\nabla\cdot\left[e^{\alpha P(x)}\nabla\psi_1\right]\mathrm{d}x.
\end{aligned}$$

This together with the equations satisfied by ϕ_1 and ψ_1 gives the equation (3.9). Likewise, by using the equality

$$\nabla \cdot \left[e^{\alpha P(x)} \phi_1^2 \nabla \left(\frac{\psi_1}{\phi_1} \right) \right] = e^{\alpha P(x)} \phi_1 (\nabla P \cdot \nabla \psi_1 + \Delta \psi_1) - e^{\alpha P(x)} \psi_1 (\nabla P \cdot \nabla \phi_1 + \Delta \phi_1)$$
$$= \phi_1 \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi_1 \right] - \psi_1 \nabla \cdot \left[e^{\alpha P(x)} \nabla \phi_1 \right],$$

we can derive the equation (3.10). The proof is completed. □

In what follows, we state the main result of this section.

Theorem 3.4. *Assume that the condition* **(H1)** *holds. We have the following conclusions:*

(i) Suppose that either $\sup_\Omega f(x,0) > 0$ *or* $\sup_{\partial\Omega} \beta(x,0) > 0$. *If*

$$\int_\Omega e^{\alpha P(x)} f(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,0)\mathrm{d}S < 0,$$

then the trivial steady state $u = 0$ *of* (1.5) *is locally asymptotically stable for* $0 < \lambda < \lambda_1$ *and unstable for* $\lambda > \lambda_1$, *where* λ_1 *is the positive principal eigenvalue of* (2.3) *with* $g = f(x,0)$ *and* $h = \beta(x,0)$.

(ii) Suppose that either $f(x,0)$ *changes sign in* Ω *or* $\beta(x,0)$ *changes sign on* $\partial\Omega$. *If*

$$\int_\Omega e^{\alpha P(x)} f(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,0)\mathrm{d}S \geq 0,$$

then the trivial steady state $u = 0$ *of* (1.5) *is always unstable.*

Proof. (i) Consider the case $\int_\Omega e^{\alpha P(x)} f(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,0)\mathrm{d}S < 0$. It follows from Theorem 2.2 (i) that (2.3) with $g = f(x,0)$ and $h = \beta(x,0)$ has a positive principal eigenvalue λ_1. By the variational characterization of λ_1, we see that $\int_\Omega e^{\alpha P(x)} f(x,0)\phi_1^2\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,0)\phi_1^2\mathrm{d}S > 0$. Inserting $\hat{u} = 0$ into (3.9), we obtain

$$\mu_1(\lambda,0) \left(\int_\Omega e^{\alpha P(x)} \phi_1^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \phi_1^2 \mathrm{d}S \right)$$
$$= - \int_\Omega e^{\alpha P(x)} \psi_1^2 \left| \nabla \left(\frac{\phi_1}{\psi_1} \right) \right|^2 \mathrm{d}x$$
$$+ (\lambda_1 - \lambda) \left(\int_\Omega e^{\alpha P(x)} f(x,0) \phi_1^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,0) \phi_1^2 \mathrm{d}S \right),$$

which implies that $\mu_1(\lambda,0) < 0$ for $\lambda > \lambda_1$.

In the following, we prove that $\mu_1(\lambda,0) > 0$ if $0 < \lambda < \lambda_1$. By setting $\hat{u} = 0$ in (3.10), it holds that

$$
\begin{aligned}
&\mu_1(\lambda,0)\left(\int_\Omega e^{\alpha P(x)}\psi_1(\lambda,0)^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\psi_1(\lambda,0)^2\mathrm{d}S\right)\\
&=\int_\Omega e^{\alpha P(x)}\phi_1^2\left|\nabla\left(\frac{\psi_1(\lambda,0)}{\phi_1}\right)\right|^2\mathrm{d}x\\
&\quad+(\lambda_1-\lambda)\left(\int_\Omega e^{\alpha P(x)}f(x,0)\psi_1(\lambda,0)^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\psi_1(\lambda,0)^2\mathrm{d}S\right).
\end{aligned}
$$

This shows that $\mu_1(\lambda,0)>0$ when $0<\lambda<\lambda_1$ and

$$
\int_\Omega e^{\alpha P(x)}f(x,0)\psi_1(\lambda,0)^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\psi_1(\lambda,0)^2\mathrm{d}S>0.
$$

Meanwhile, for the case $0<\lambda<\lambda_1$ and

$$
\int_\Omega e^{\alpha P(x)}f(x,0)\psi_1(\lambda,0)^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\psi_1(\lambda,0)^2\mathrm{d}S\le 0,
$$

by using (3.1) with $\hat{u}=0$, we have

$$
\begin{aligned}
&\mu_1(\lambda,0)\left(\int_\Omega e^{\alpha P(x)}\psi_1(\lambda,0)^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\psi_1(\lambda,0)^2\mathrm{d}S\right)\\
&=\int_\Omega e^{\alpha P(x)}\left|\nabla\psi_1(\lambda,0)\right|^2\mathrm{d}x\\
&\quad-\lambda\left(\int_\Omega e^{\alpha P(x)}f(x,0)\psi_1(\lambda,0)^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\psi_1(\lambda,0)^2\mathrm{d}S\right)\\
&\ge\int_\Omega e^{\alpha P(x)}\left|\nabla\psi_1(\lambda,0)\right|^2\mathrm{d}x.
\end{aligned}
$$

Since $\psi_1(\lambda,0)$ is not a constant for all $\lambda>0$, there must be $\mu_1(\lambda,0)>0$. This completes the proof of part (i).

(ii) Consider the case $\int_\Omega e^{\alpha P(x)}f(x,0)\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\mathrm{d}S\ge 0$. Then we see from Theorem 2.2 (ii) and (iii) that the unique nonnegative principal eigenvalue of (2.3) is $\lambda_1=0$ with a eigenfunction $\phi_1=1$. By setting $\hat{u}=0$ in (3.9), we obtain

$$
\mu_1(\lambda,0)\left(\int_\Omega e^{\alpha P(x)}\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\mathrm{d}S\right)
$$

$$= -\int_{\Omega} e^{\alpha P(x)} \psi_1(\lambda,0)^2 \left|\nabla\left(\frac{1}{\psi_1(\lambda,0)}\right)\right|^2 \mathrm{d}x$$
$$-\lambda\left(\int_{\Omega} e^{\alpha P(x)} f(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\mathrm{d}S\right).$$

Since $\psi_1(\lambda,0)$ is not a constant for all $\lambda > 0$, we have $\mu_1(\lambda,0) < 0$. Thanks to Proposition 3.2, we complete the proof of this theorem. □

4. Bifurcation analysis at trivial steady states

This section is devoted to the bifurcations of non-trivial solutions of (1.6) from two lines of trivial solutions Γ_0 and Γ_{u_1} defined in (1.7) and (1.8), respectively. In a special situation, non-trivial solutions can also bifurcate from $(\lambda,u) = (0,0)$, the intersection point of Γ_0 and Γ_{u_1}.

Definition 4.1. A point $(\lambda_*,0)$ is called a bifurcation point on the line of trivial solutions Γ_0 if there exists a sequence of solutions $(\lambda^{(n)},u^{(n)})$ to (1.6) such that $u^{(n)} \neq 0, \lambda^{(n)} \to \lambda_*$ and $u^{(n)} \to 0$ in $C(\overline{\Omega})$ as $n \to \infty$. And a bifurcation point on the line of trivial solutions Γ_{u_1} is defined similarly.

To consider the solutions of (1.6), we define a nonlinear mapping $\mathcal{F}: \mathbb{R}\times X \to Y$ by

$$\mathcal{F}(\lambda,u) = \Big(\nabla\cdot\big[e^{\alpha P(x)}\nabla u\big] + \lambda e^{\alpha P(x)} f(x,u)u,\ \frac{\partial u}{\partial \vec{n}} - \lambda\beta(x,u)u\Big). \tag{4.1}$$

Note that the Fréchet derivative $D_u\mathcal{F}(\lambda_*,u_*): X \to Y$ of $\mathcal{F}$ with respect to u at (λ_*,u_*) is given by

$$\begin{aligned} D_u\mathcal{F}(\lambda_*,u_*)[v] = \Big(&\nabla\cdot\big[e^{\alpha P(x)}\nabla v\big] + \lambda_* e^{\alpha P(x)}\big(f(x,u_*) + f_u(x,u_*)u_*\big)v,\\ &\frac{\partial v}{\partial \vec{n}} - \lambda_*\big(\beta(x,u_*) + \beta_u(x,u_*)u_*\big)v\Big). \end{aligned} \tag{4.2}$$

It is well known that if (λ_*,u_*) is a bifurcation point on the line Γ_0 (or Γ_{u_1}), then the operator $D_u\mathcal{F}(\lambda_*,u_*)$ is not injective. We first provide the following lemma concerning the potential bifurcation points.

Lemma 4.2. *Assume that the condition* **(H1)** *holds.*

(i) If $(\lambda_*,0)$ *with* $\lambda_* > 0$ *is a bifurcation point of* (1.6) *with respect to the trivial branch* Γ_0, *then* λ_* *is an eigenvalue of*

$$\begin{cases} -\nabla\cdot\big[e^{\alpha P(x)}\nabla v\big] = \lambda e^{\alpha P(x)} f(x,0)v, & x\in\Omega,\\ \partial_{\vec{n}} v = \lambda\beta(x,0)v, & x\in\partial\Omega. \end{cases} \tag{4.3}$$

(ii) If $(0,u_*)$ *with* $u_* > 0$ *is a bifurcation point of* (1.6) *with respect to the trivial branch* Γ_{u_1}, *then* u_* *satisfies*

(H2) $$\int_{\Omega} e^{\alpha P(x)} f(x, u_*) \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x, u_*) \mathrm{d}S = 0.$$

Proof. (i) Suppose on the contrary that λ_* is not an eigenvalue of (4.3). Then $D_u\mathcal{F}(\lambda_*, 0)$ is a homeomorphism for all λ near λ_*. In view of the implicit function theorem, we see that the trivial solution $(\lambda_*, 0)$ is the unique solution of $\mathcal{F} = 0$ near $(\lambda_*, 0)$. Then λ_* is not a bifurcation point along the line Γ_0. Hence λ_* must be an eigenvalue of (4.3) if $(\lambda_*, 0)$ is a bifurcation point.

(ii) Suppose that $(0, u_*)$ with $u_* > 0$ is a bifurcation point of (1.6) with respect to the trivial branch Γ_{u_1}. Then from Definition 4.1, there exists a sequence $(\lambda^{(n)}, u^{(n)})$ of solutions to (1.6) with

$$0 \neq \lambda^{(n)} \to 0 \ \text{ and } \ \|u^{(n)} - u_*\|_X \to 0, \quad \text{when } n \to \infty.$$

A integration on the equation satisfied by $u^{(n)}$ yields that

$$\begin{aligned}\lambda^{(n)} \int_{\Omega} e^{\alpha P(x)} f(x, u^{(n)}) u^{(n)} \mathrm{d}x &= -\int_{\Omega} \nabla \cdot \left[e^{\alpha P(x)} \nabla u\right] = -\int_{\partial\Omega} e^{\alpha P(x)} \frac{\partial u}{\partial \vec{n}} \mathrm{d}S \\ &= -\lambda^{(n)} \int_{\partial\Omega} e^{\alpha P(x)} \beta(x, u^{(n)}) u^{(n)} \mathrm{d}S,\end{aligned}$$

which leads to that

$$\int_{\Omega} e^{\alpha P(x)} f(x, u^{(n)}) u^{(n)} \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x, u^{(n)}) u^{(n)} \mathrm{d}S = 0$$

since $\lambda^{(n)} \neq 0$. Consequently, we obtain **(H2)** by taking $n \to \infty$. □

4.1. Local bifurcation from Γ_0

In this subsection, we consider the bifurcations on $\Gamma_0 = \{(\lambda, 0) : \lambda > 0\}$. From Theorem 2.2, we see that under the condition

$$\int_{\Omega} e^{\alpha P(x)} f(x, 0) \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x, 0) \mathrm{d}S < 0,$$

$(\lambda_1, 0)$ is a potential bifurcation point of (1.6) on the line of trivial solution Γ_0, where λ_1 is the unique positive principal eigenvalue of (4.3), which is always a simple one. Then we have the following result on the local bifurcation from Γ_0.

Theorem 4.3. *Assume that f, β satisfy* **(H1)** *and also satisfy*

(H3) *Either* $\sup_{\Omega} f(x, 0) > 0$ *or* $\sup_{\partial\Omega} \beta(x, 0) > 0$, *and*

$$\int_{\Omega} e^{\alpha P(x)} f(x, 0) \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x, 0) \mathrm{d}S < 0. \tag{4.4}$$

Then the eigenvalue problem (4.3) *admits a positive principal eigenvalue* λ_1*, which is a bifurcation point with respect to* Γ_0*. Precisely, in a neighborhood of* $(\lambda_1, 0)$ *in* $\mathbb{R}^+ \times X$*, the only positive solution to* (1.6) *lies in the curve*

$$\Sigma_0 = \{(\lambda_0(s), u_0(s)) : s \in I = (0, \varepsilon) \subset \mathbb{R}^+\},$$

where $\lambda_0(s) = \lambda_1 + z_2(s)$*,* $u_0(s) = s\phi_1 + sz_1(s)$ *are* C^1 *function so that* $z_i(0) = 0, i = 1, 2,$ ϕ_1 *is the positive eigenfunction associated with* λ_1*.*

Proof. Under the assumptions **(H1)** and **(H3)**, we see from Theorem 2.2 (i) that the problem (4.3) has a principal eigenvalue $\lambda_1 > 0$. We apply Crandall-Rabinowitz's Theorem [20] for simple eigenvalue to prove our assertion. For this proof, we will find the nontrivial solution of $\mathcal{F} = 0$ near $(\lambda_1, 0)$, where $\mathcal{F}$ is defined in (4.1). Now, we complete it by several steps:

(a) $D_\lambda\mathcal{F}$, $D_u\mathcal{F}$ and $D_{\lambda u}\mathcal{F}$ exist and are continuous. This assertion is obvious.
(b) $\dim N(D_u\mathcal{F}(\lambda_1, 0)) = 1$. By Theorem 2.2 (i), the positive principal eigenvalue λ_1 of (4.3) is simple, and ϕ_1 is a positive eigenfunction associated with λ_1. Thus, $N(D_u\mathcal{F}(\lambda_1, 0)) = \mathrm{span}\{\phi_1\}$.
(c) $\mathrm{codim} R(D_u\mathcal{F}(\lambda_1, 0)) = 1$. Set $(y_1, y_2) \in R(D_u\mathcal{F}(\lambda_1, 0))$ and $w \in X$ satisfying that

$$\begin{cases} \nabla \cdot \left[e^{\alpha P(x)}\nabla w\right] + \lambda_1 e^{\alpha P(x)} f(x, 0)w = y_1, & x \in \Omega, \\ \partial_{\vec{n}} w - \lambda_1 \beta(x, 0)w = y_2, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

Multiplying the first equation of (4.3) and (4.5) by w and ϕ_1, respectively, subtracting and integrating the result over Ω, we obtain

$$\begin{aligned} \int_\Omega \phi_1 y_1 \mathrm{d}x &= \int_\Omega \left(\phi_1 \nabla \cdot \left[e^{\alpha P(x)}\nabla w\right] - w\nabla \cdot \left[e^{\alpha P(x)}\nabla\phi_1\right]\right)\mathrm{d}x \\ &= \int_{\partial\Omega} e^{\alpha P(x)}(\phi_1 \frac{\partial w}{\partial \vec{n}} - w\frac{\partial \phi_1}{\partial \vec{n}})\mathrm{d}S \\ &= \int_{\partial\Omega} e^{\alpha P(x)}(\phi_1(\lambda_1\beta(x, 0)w + y_2) - w\lambda_1\beta(x, 0)\phi_1)\mathrm{d}S = \int_{\partial\Omega} e^{\alpha P(x)}\phi_1 y_2 \mathrm{d}S. \end{aligned}$$

Thus, $(y_1, y_2) \in R(D_u\mathcal{F}(\lambda_1, 0))$ if and only if

$$\int_\Omega \phi_1 y_1 \mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)}\phi_1 y_2 \mathrm{d}S = 0. \tag{4.6}$$

Thus, $\mathrm{codim} R(D_u\mathcal{F}(\lambda_1, 0)) = 1$. In the following we define $l \in Y^*$ as

$$\langle l, (y_1, y_2)\rangle = \int_\Omega \phi_1 y_1 \mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)}\phi_1 y_2 \mathrm{d}S. \tag{4.7}$$

(d) $D_{\lambda u}\mathcal{F}(\lambda_1, 0)[\phi_1] \notin R(D_u\mathcal{F}(\lambda_1, 0))$. A direct calculation gives that

$$D_{\lambda u}\mathcal{F}(\lambda_1, 0)[\phi_1] = \big(e^{\alpha P(x)} f(x, 0)\phi_1, -\beta(x, 0)\phi_1\big).$$

However, we see that

$$\begin{aligned}
&\int_\Omega e^{\alpha P(x)} f(x, 0)\phi_1^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\beta(x, 0)\phi_1^2 \mathrm{d}S \\
=&-\frac{1}{\lambda_1}\int_\Omega \phi_1 \nabla\cdot\big[e^{\alpha P(x)}\nabla\phi_1\big]\mathrm{d}x + \frac{1}{\lambda_1}\int_{\partial\Omega} e^{\alpha P(x)}\frac{\partial\phi_1}{\partial\vec{n}}\phi_1 \mathrm{d}S \\
=&\frac{1}{\lambda_1}\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x > 0.
\end{aligned} \tag{4.8}$$

This verifies that $D_{\lambda u}\mathcal{F}(\lambda_1, 0)[\phi_1] \notin R(D_u\mathcal{F}(\lambda_1, 0))$.

Hence this theorem follows by applying [20, Theorem 1.7]. □

Remark 4.4. We can see from the proof of Theorem 4.3 that there exists a negative solution curve Σ_0^- of (1.6) bifurcating from the line of trivial solutions Γ_0, which has the form $\Sigma_0^- = \{(\lambda_0(s), u_0(s)) : s \in I = (-\eta, 0) \subset \mathbb{R}\}$. However, this solution has no biological significance.

In the following, we investigate the bifurcation direction of the steady state bifurcation from $(\lambda_1, 0)$ derived in Theorem 4.3. For the purpose, we need to calculating the sign of $\lambda_0'(0)$. Assume that f, β are class C^2 or C^3 near $u = 0$. Using the expression (4.5) in [53], we have

$$\lambda_0'(0) = -\frac{\langle l, D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1]^2\rangle}{2\langle l, D_{\lambda u}\mathcal{F}(\lambda_1, 0)[\phi_1]\rangle}. \tag{4.9}$$

And the results in [53] can be concluded that

(F1) if $\mathcal{F}$ satisfies $D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1]^2 \notin R(D_u\mathcal{F}(\lambda_1, 0))$, then $\lambda_0'(0) \neq 0$, and it is called a transcritical bifurcation;

(F2) if $\mathcal{F}$ satisfies $D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1]^2 \in R(D_u\mathcal{F}(\lambda_1, 0))$, then $\lambda_0'(0) = 0$, and

$$\lambda_0''(0) = -\frac{\langle l, D_{uuu}\mathcal{F}(\lambda_1, 0)[\phi_1]^3\rangle + 3\langle l, D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1, \vartheta]\rangle}{3\langle l, D_{\lambda u}\mathcal{F}(\lambda_1, 0)[\phi_1]\rangle}, \tag{4.10}$$

where ϑ satisfies $D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1]^2 + D_u\mathcal{F}(\lambda_1, 0)[\vartheta] = 0$. Moreover, if $\lambda_0'(0) = 0$ and $\lambda_0''(0) \neq 0$, then it is called a pitchfork bifurcation.

A direct calculation gives that

$$\begin{aligned}
D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1]^2 &= \big(2\lambda_1 e^{\alpha P(x)} f_u(x, 0)\phi_1^2, -2\lambda_1\beta_u(x, 0)\phi_1^2\big), \\
D_{uu}\mathcal{F}(\lambda_1, 0)[\phi_1, \vartheta] &= \big(2\lambda_1 e^{\alpha P(x)} f_u(x, 0)\phi_1\vartheta, -2\lambda_1\beta_u(x, 0)\phi_1\vartheta\big)
\end{aligned}$$

and

$$D_{uuu}\mathcal{F}(\lambda_1,0)[\phi_1]^3=\big(3\lambda_1 e^{\alpha P(x)}f_{uu}(x,0)\phi_1^3,-3\lambda_1\beta_{uu}(x,0)\phi_1^3\big).$$

Then we have

$$\lambda_0'(0)=-\frac{\lambda_1^2\int_\Omega e^{\alpha P(x)}f_u(x,0)\phi_1^3\mathrm{d}x+\lambda_1^2\int_{\partial\Omega}e^{\alpha P(x)}\beta_u(x,0)\phi_1^3\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}$$

and

$$\begin{aligned}\lambda_0''(0)=&-\frac{\lambda_1^2\int_\Omega e^{\alpha P(x)}f_{uu}(x,0)\phi_1^4\mathrm{d}x+\lambda_1^2\int_{\partial\Omega}e^{\alpha P(x)}\beta_{uu}(x,0)\phi_1^4\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&-\frac{2\lambda_1^2\int_\Omega e^{\alpha P(x)}f_u(x,0)\phi_1^2\vartheta\mathrm{d}x+2\lambda_1^2\int_{\partial\Omega}e^{\alpha P(x)}\beta_u(x,0)\phi_1^2\vartheta\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x},\end{aligned}$$

where ϑ satisfies

$$\begin{cases}\nabla\cdot\big[e^{\alpha P(x)}\nabla\vartheta\big]+\lambda_1 e^{\alpha P(x)}f(x,0)\vartheta+2\lambda_1 e^{\alpha P(x)}f_u(x,0)\phi_1^2=0, & x\in\Omega,\\ \partial_{\vec{n}}\vartheta-\lambda_1\beta(x,0)\vartheta-2\lambda_1\beta_u(x,0)\phi_1^2=0, & x\in\partial\Omega.\end{cases}\tag{4.11}$$

The above expression for $\lambda_0'(0)$ can be simplified in several special cases, which are listed as follows:

(Case 1) Assume that $f(x,u)\equiv 0$ and $\beta(x,u)=r(x)b(u)$, where $r(x)\not\equiv 0$ and $b(0)\neq 0$. Then

$$\begin{aligned}\lambda_0'(0)&=-\frac{\lambda_1^2 b'(0)\int_{\partial\Omega}e^{\alpha P(x)}r(x)\phi_1^3\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}=-\lambda_1\cdot\frac{b'(0)}{b(0)}\cdot\frac{\int_{\partial\Omega}e^{\alpha P(x)}\phi_1^2\partial_{\vec{n}}\phi_1\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=-\lambda_1\cdot\frac{b'(0)}{b(0)}\cdot\frac{\int_\Omega\phi_1^2\nabla\cdot\big[e^{\alpha P(x)}\nabla\phi_1\big]\mathrm{d}x+2\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=-2\lambda_1\cdot\frac{b'(0)}{b(0)}\cdot\frac{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}.\end{aligned}\tag{4.12}$$

Thus if $b'(0)\neq 0$, then a transcritical bifurcation occurs at $(\lambda_1,0)$, while if $b'(0)=0$, then $\lambda_0'(0)=0$. For the case $b'(0)=0$, if we further assume that $b\in C^3$ near $u=0$, then

$$\begin{aligned}\lambda_0''(0)&=-\frac{\lambda_1^2 b''(0)\int_{\partial\Omega}e^{\alpha P(x)}r(x)\phi_1^4\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}=-\lambda_1\cdot\frac{b''(0)}{b(0)}\cdot\frac{\int_{\partial\Omega}e^{\alpha P(x)}\phi_1^3\partial_{\vec{n}}\phi_1\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=-\lambda_1\cdot\frac{b''(0)}{b(0)}\cdot\frac{\int_\Omega\phi_1^3\nabla\cdot\big[e^{\alpha P(x)}\nabla\phi_1\big]\mathrm{d}x+3\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1^2\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=-3\lambda_1\cdot\frac{b''(0)}{b(0)}\cdot\frac{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1^2\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}.\end{aligned}\tag{4.13}$$

Thus, if $b'(0)=0$ and $b''(0)\neq 0$, then a pitchfork bifurcation occurs at $(\lambda_1,0)$.

(Case 2) Assume that $\beta(x,u)\equiv 0$ and $f(x,u)=k(x)a(u)$, where $k(x)\not\equiv 0$ and $a(0)\neq 0$. Then

$$\begin{aligned}\lambda_0'(0)&=-\frac{\lambda_1^2 a'(0)\int_\Omega e^{\alpha P(x)}k(x)\phi_1^3\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}=\lambda_1\cdot\frac{a'(0)}{a(0)}\cdot\frac{\int_\Omega \phi_1^2\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=\lambda_1\cdot\frac{a'(0)}{a(0)}\cdot\frac{\int_{\partial\Omega}\phi_1^2 e^{\alpha P(x)}\partial_{\vec{n}}\phi_1\mathrm{d}x-2\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=-2\lambda_1\cdot\frac{a'(0)}{a(0)}\cdot\frac{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}.\end{aligned}\tag{4.14}$$

Thus if $a'(0)\neq 0$, then a transcritical bifurcation occurs at $(\lambda_1,0)$, while if $a'(0)=0$, then $\lambda_0'(0)=0$. For the case $a'(0)=0$, if $a\in C^3$ near $u=0$, then

$$\begin{aligned}\lambda_0''(0)&=-\frac{\lambda_1^2 a''(0)\int_\Omega e^{\alpha P(x)}k(x)\phi_1^4\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}=\lambda_1\cdot\frac{a''(0)}{a(0)}\cdot\frac{\int_\Omega \phi_1^3\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=\lambda_1\cdot\frac{a''(0)}{a(0)}\cdot\frac{\int_{\partial\Omega}\phi_1^3 e^{\alpha P(x)}\partial_{\vec{n}}\phi_1\mathrm{d}S-3\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1^2\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=-3\lambda_1\cdot\frac{a''(0)}{a(0)}\cdot\frac{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1^2\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}.\end{aligned}\tag{4.15}$$

Therefore, if $a'(0)=0$ and $a''(0)\neq 0$, then a pitchfork bifurcation occurs at $(\lambda_1,0)$.

(Case 3) Assume that $f(x,u)=k(x)a(u)$ and $\beta(x,u)=r(x)b(u)$, where $k(x)\not\equiv 0, r(x)\not\equiv 0$ and $a(0)\neq 0, b(0)\neq 0$. Then

$$\begin{aligned}\lambda_0'(0)&=-\frac{\lambda_1^2 a'(0)\int_\Omega e^{\alpha P(x)}k(x)\phi_1^3\mathrm{d}x+\lambda_1^2 b'(0)\int_{\partial\Omega}e^{\alpha P(x)}r(x)\phi_1^3\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=\lambda_1\cdot\frac{a'(0)}{a(0)}\cdot\frac{\int_\Omega \phi_1^2\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}-\lambda_1\cdot\frac{b'(0)}{b(0)}\cdot\frac{\int_{\partial\Omega}e^{\alpha P(x)}\phi_1^2\partial_{\vec{n}}\phi_1\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=\lambda_1\cdot\left(\frac{a'(0)}{a(0)}-\frac{b'(0)}{b(0)}\right)\cdot\frac{\int_\Omega \phi_1^2\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&\quad-2\lambda_1\cdot\frac{b'(0)}{b(0)}\cdot\frac{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\phi_1\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}.\end{aligned}\tag{4.16}$$

If $a(0)b'(0)=b(0)a'(0)\neq 0$, then (4.16) becomes (4.12) and a transcritical bifurcation occurs. While if $a'(0)=b'(0)=0$, then $\lambda_0'(0)=0$ and

$$\begin{aligned}\lambda_0''(0)&=-\frac{\lambda_1^2 a''(0)\int_\Omega e^{\alpha P(x)}k(x)\phi_1^4\mathrm{d}x+\lambda_1^2 b''(0)\int_{\partial\Omega}e^{\alpha P(x)}r(x)\phi_1^4\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\\&=\lambda_1\cdot\frac{a''(0)}{a(0)}\cdot\frac{\int_\Omega \phi_1^3\nabla\cdot\left[e^{\alpha P(x)}\nabla\phi_1\right]\mathrm{d}x}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}-\lambda_1\cdot\frac{b''(0)}{b(0)}\cdot\frac{\int_{\partial\Omega}e^{\alpha P(x)}\phi_1^3\partial_{\vec{n}}\phi_1\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}|\nabla\phi_1|^2\mathrm{d}x}\end{aligned}$$

$$= \lambda_1 \cdot \left(\frac{a''(0)}{a(0)} - \frac{b''(0)}{b(0)} \right) \cdot \frac{\int_\Omega \phi_1^3 \nabla \cdot \left[e^{\alpha P(x)} \nabla \phi_1 \right] \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} |\nabla \phi_1|^2 \mathrm{d}x} - 3\lambda_1 \cdot \frac{b''(0)}{b(0)} \cdot \frac{\int_\Omega e^{\alpha P(x)} |\nabla \phi_1|^2 \phi_1^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} |\nabla \phi_1|^2 \mathrm{d}x}. \tag{4.17}$$

We find that the sign of $\lambda_0'(0)$ or $\lambda_0''(0)$ in the formulas (4.12), (4.13), (4.14) and (4.15) is independent of the weighted function $k(x)$ or $r(x)$. However, the calculation for (4.16) or (4.17) is more complicated due to the weighted function $k(x)$ or $r(x)$.

Finally we discuss the stability of the positive bifurcating solution $(\lambda_0(s), u_0(s))$ derived in Theorem 4.3. Consider the linear eigenvalue problem (3.1) at $(\lambda_0(s), u_0(s))$, which read as:

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi \right] - \lambda_0(s) \left[f(x, u_0(s)) + u_0(s) f_u(x, u_0(s)) \right] \psi = \mu \psi, & x \in \Omega, \\ \partial_{\vec{n}} \psi - \lambda_0(s) \left[\beta(x, u_0(s)) + u_0(s) \beta_u(x, u_0(s)) \right] \psi = \mu \psi, & x \in \partial\Omega. \end{cases} \tag{4.18}$$

To stress the dependence of the principal eigenvalue of (4.18) on s, we denote by $\mu_1(s)$ the principal eigenvalue of problem (4.18), and let $\psi_1(s)$ be the positive eigenfunction corresponding to $\mu_1(s)$. It is clear that $\lambda_0(0) = \lambda_1, u_0(0) = 0, \mu_1(0) = 0$ and $\psi_1(0) = \phi_1$. Moreover, $\mu_1(s)$ and $\psi_1(s)$ are both real analytic at $s = 0$. Indeed, looking at the mapping $G : (-\varepsilon, \varepsilon) \times \mathbb{R} \times W_l^2(\Omega) \to L^l(\Omega) \times W_l^{1-\frac{1}{l}}(\partial\Omega) \times \mathbb{R}$, where $l > N$, defined by

$$G(s, \mu, \psi) = \begin{pmatrix} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi \right] - \lambda_0(s) \left[f(x, u_0(s)) + u_0(s) f_u(x, u_0(s)) \right] \psi - \mu \psi \\ \partial_{\vec{n}} \psi - \lambda_0(s) \left[\beta(x, u_0(s)) + u_0(s) \beta_u(x, u_0(s)) \right] \psi - \mu \psi \\ \int_\Omega e^{\alpha P(x)} \psi^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \psi^2 \mathrm{d}S - \left(\int_\Omega e^{\alpha P(x)} \phi_1^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \phi_1^2 \mathrm{d}S \right) \end{pmatrix},$$

we have that $G(0, 0, \phi_1) = 0$ and the Fréchet derivative $D_{(\mu,\psi)} G(0, 0, \phi_1) : (-\varepsilon, \varepsilon) \times \mathbb{R} \times W_l^2(\Omega) \to L^l(\Omega) \times W_l^{1-\frac{1}{l}}(\partial\Omega) \times \mathbb{R}$ with respect to (μ, ψ) at $(0, 0, \phi_1)$ is a homeomorphism by the standard argument. Then an application of the implicit function theorem implies that

$$G(s, \mu, \psi) = 0 \Longleftrightarrow (\mu, \psi) = (\mu(s), \psi(s)), \quad |s| \ll 1,$$

$\mu(0) = 0, \psi(0) = \phi_1$ and $\mu(s)$ and $\psi(s)$ are both real analytic at $s = 0$ (see Zeidler [64]). Here $\psi(s)$ is positive on $\overline{\Omega}$ by the positivity of ϕ_1 on $\overline{\Omega}$ and the continuity of $\psi(s)$ on s. Hence we obtain that $\mu(s) = \mu_1(s)$ and $\psi(s) = \psi_1(s)$ by uniqueness.

Now we differentiate by s the equation (4.18) with $\mu = \mu_1(s)$ and $\psi = \psi_1(s)$, and then obtain the equation for $s = 0$:

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi_1'(0) \right] \\ \quad -\lambda_1 f(x, 0) \psi_1'(0) - \lambda_0'(0) f(x, 0) \phi_1 - 2\lambda_1 f_u(x, 0) \phi_1^2 = \mu_1'(0) \phi_1, & x \in \Omega, \\ \partial_{\vec{n}} \psi_1'(0) - \lambda_1 \beta(x, 0) \psi_1'(0) - \lambda_0'(0) \beta(x, 0) \phi_1 - 2\lambda_1 \beta_u(x, 0) \phi_1^2 = \mu_1'(0) \phi_1, & x \in \partial\Omega, \end{cases} \tag{4.19}$$

where we have used the fact that $\lambda_0(0) = \lambda_1, u_0(0) = 0, \mu_1(0) = 0$ and $u_0'(0) = \psi_1(0) = \phi_1$. Multiplying the first equation of (4.19) by $e^{\alpha P(x)} \phi_1$ and integrating the result over Ω, it follows that

$$
\begin{aligned}
&\mu_1'(0)\Big(\int_\Omega e^{\alpha P(x)}\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\phi_1^2\mathrm{d}S\Big)\\
&\quad=-\lambda_0'(0)\Big(\int_\Omega e^{\alpha P(x)}f(x,0)\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\phi_1^2\mathrm{d}S\Big)\\
&\qquad-2\lambda_1\Big(\int_\Omega e^{\alpha P(x)}f_u(x,0)\phi_1^3\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta_u(x,0)\phi_1^3\mathrm{d}S\Big).
\end{aligned}\tag{4.20}
$$

This combined with the expression of $\lambda_0'(0)$ and the variational characterization of λ_1 gives that

$$
\mu_1'(0)=\frac{\lambda_0'(0)\left(\int_\Omega e^{\alpha P(x)}f(x,0)\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\phi_1^2\mathrm{d}S\right)}{\int_\Omega e^{\alpha P(x)}\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\phi_1^2\mathrm{d}S}.\tag{4.21}
$$

Thanks to the variational characterization of λ_1, it follows that

$$
\int_\Omega e^{\alpha P(x)}f(x,0)\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\phi_1^2\mathrm{d}S>0.
$$

Furthermore, when $\lambda_0'(0)=0$, we still need to compute the expression of $\mu_1''(0)$ since $\mu_1'(0)=0$. Differentiating twice by s the equation (4.18) with $\mu=\mu_1(s)$ and $\psi=\psi_1(s)$ and letting $s=0$, similarly, we have

$$
\begin{aligned}
&\mu_1''(0)\left(\int_\Omega e^{\alpha P(x)}\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\phi_1^2\mathrm{d}S\right)\\
&=-4\lambda_1\left(\int_\Omega e^{\alpha P(x)}f_u(x,0)\phi_1^2\psi_1'(0)\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta_u(x,0)\phi_1^2\psi_1'(0)\mathrm{d}S\right)\\
&\quad-\lambda_0''(0)\left(\int_\Omega e^{\alpha P(x)}f(x,0)\phi_1^2\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\phi_1^2\mathrm{d}S\right)\\
&\quad-\lambda_1\Bigg(\int_\Omega e^{\alpha P(x)}\big[3f_{uu}(x,0)\phi_1^2+2f_u(x,0)u_0''(0)\big]\phi_1^2\mathrm{d}x\\
&\quad+\int_{\partial\Omega} e^{\alpha P(x)}\big[3\beta_{uu}(x,0)\phi_1^2+2\beta_u(x,0)u_0''(0)\big]\phi_1^2\mathrm{d}S\Bigg).
\end{aligned}\tag{4.22}
$$

In (4.22), there are $u_0''(0)$ and $\psi_1'(0)$ to be determined. Here, $\psi_1'(0)$ satisfies (4.19) with $\lambda_0'(0)=\mu_1'(0)=0$, which coincides with (4.11). To consider $u_0''(0)$, we substitute $(\lambda,u)=(\lambda_0(s),s\phi_1+sz_1(s))$ into (1.6), and then obtain

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla(\phi_1 + z_1(s))\right] = \lambda_0(s) e^{\alpha P(x)} f(x, s\phi_1 + s z_1(s))(\phi_1 + z_1(s)), & x \in \Omega, \\ \partial_{\vec{n}}(\phi_1 + z_1(s)) = \lambda_0(s) \beta(x, s\phi_1 + s z_1(s))(\phi_1 + z_1(s)), & x \in \partial\Omega. \end{cases} \tag{4.23}$$

By differentiating (4.23) with respect to s, and letting $s = 0$, there holds that

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla z_1'(0)\right] = \lambda_1 e^{\alpha P(x)} f(x, 0) z_1'(0) + \lambda_1 e^{\alpha P(x)} f_u(x, 0)\phi_1^2, & x \in \Omega, \\ \partial_{\vec{n}} z_1'(0) = \lambda_1 \beta(x, 0) z_1'(0) + \lambda_1 \beta_u(x, 0)\phi_1^2, & x \in \partial\Omega. \end{cases} \tag{4.24}$$

Note that $u_0''(s) = 2z_1'(s) + s z_1''(s)$. Then $u_0''(0) = 2z_1'(0)$. According to definitions of $\psi_1(s)$ and $u_0(s)$, we see that $\psi_1'(0)$, $u_0''(0)$ and $z_1'(0)$ do not belong to $N(D_u\mathcal{F}(\lambda_1, 0))$. Notice that

$$\int_{\Omega} e^{\alpha P(x)} f_u(x, 0)\phi_1^3 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta_u(x, 0)\phi_1^3 \mathrm{d}S = 0$$

since $\lambda_0'(0) = 0$. Then by (4.6), the solution of (4.11) exists and is unique, so does the solution of (4.24). Now, we must have $\psi_1'(0) = u_0''(0) = \vartheta$, where ϑ is defined as in (4.11). This together with (4.22) and the expression of $\lambda_0''(0)$ yields that

$$\mu_1''(0) = \frac{2\lambda_0''(0) \left(\int_\Omega e^{\alpha P(x)} f(x, 0)\phi_1^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x, 0)\phi_1^2 \mathrm{d}S\right)}{\int_\Omega e^{\alpha P(x)}\phi_1^2 \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\phi_1^2 \mathrm{d}S}. \tag{4.25}$$

To sum up the above discussion, we have that under the assumptions **(H1)** and **(H3)**, the trivial steady state $u = 0$ is stable for $0 < \lambda < \lambda_1$, and unstable for $\lambda > \lambda_1$. If $u = 0$ and the only positive solutions are physically meaningful solutions, then the transcritical bifurcation is superscritical (resp. subscritical) and the positive bifurcating solution is stable (resp. unstable) if $\lambda_0'(0) > 0$ (resp. $\lambda_0'(0) < 0$). Furthermore, when $\lambda_0'(0) = 0$, the pitchfork bifurcation is forward (resp. backward) and the positive bifurcating solution is stable (resp. unstable) if $\lambda_0''(0) > 0$ (resp. $\lambda_0''(0) < 0$).

4.2. Local bifurcation from Γ_{u_1}

In this subsection, we discuss the bifurcations on $\Gamma_{u_1} = \{(0, u_1) : u_1 > 0, u_1 \in \mathbb{R}\}$. Notice that $\mathcal{F}(0, u_*) = 0$ and $D_u\mathcal{F}(0, u_*) = (\nabla \cdot \left[e^{\alpha P(x)}\nabla\right], \frac{\partial}{\partial \vec{n}})$ for any given constant $u_* > 0$. Then we see that $N(D_u\mathcal{F}(0, u_*)) = \mathrm{span}\{1\}$. Set $(y_1, y_2) \in R(D_u\mathcal{F}(0, u_*))$ and $w \in X$ with

$$\begin{cases} \nabla \cdot \left[e^{\alpha P(x)} \nabla w\right] = y_1, & x \in \Omega, \\ \partial_{\vec{n}} w = y_2, & x \in \partial\Omega. \end{cases}$$

Integrating the above equation over Ω, it follows that

$$\int_{\partial\Omega} e^{\alpha P(x)} y_2 \mathrm{d}S = \int_{\partial\Omega} e^{\alpha P(x)} \partial_{\vec{n}} w \mathrm{d}S = \int_{\Omega} \nabla \cdot [e^{\alpha P(x)} \nabla w] \mathrm{d}x = \int_{\Omega} y_1 \mathrm{d}x.$$

This means that $(y_1, y_2) \in R(D_u\mathcal{F}(0, u_*))$ if and only if

$$\int_{\Omega} y_1 \mathrm{d}x = \int_{\partial\Omega} e^{\alpha P(x)} y_2 \mathrm{d}S. \tag{4.26}$$

Thus, $\operatorname{codim} R(D_u\mathcal{F}(0,u_*)) = 1$ and $D_u\mathcal{F}(0,u_*)$ is a Fredholm operator with index zero. Then we decompose the spaces X and Y by

$$X = N(D_u\mathcal{F}(0,u_*)) \oplus X_1 \ \text{ and } \ Y = R(D_u\mathcal{F}(0,u_*)) \oplus Y_1.$$

In the following, referring to Section 2.2 in [26], we use the Lyapunov-Schmidt reduction technique to investigate how bifurcation occurs. When $\dim Y_1 = 1$, there exists $\varphi \in Y$ satisfying $\|\varphi\|_Y = 1$ so that $Y_1 = \operatorname{span}\{\varphi\}$. Applying the Hahn-Banach Theorem [63], we see that there exists a vector l in the dual space Y^* of Y satisfying that $\langle l, \varphi\rangle = 1$ and $\langle l, y\rangle = 0$ for all $y \in R(D_u\mathcal{F}(0,u_*))$, in which $\langle\cdot,\cdot\rangle : Y^* \times Y \to \mathbb{R}$ is the duality between Y^* and Y and is defined by

$$\langle v, u\rangle = \int_{\Omega} v(x)u_1(x)\mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)} v(x)u_2(x)\mathrm{d}S,$$

for all $v \in Y^*$ and $u = (u_1, u_2) \in Y$. Here we can choose $l \in Y^*$ such that

$$\langle l, u\rangle = \int_{\Omega} u_1(x)\mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)} u_2(x)\mathrm{d}S$$

and then $N(l) = R(D_u\mathcal{F}(0,u_*))$. Now, we let P be a projection operator from Y to Y_1 along $R(D_u\mathcal{F}(0,u_*))$, namely, $Py = \langle\psi, y\rangle\varphi$ for $y \in Y$. Set $u = u_* + \sigma + \eta$, where $\sigma \in \mathbb{R}$ and $\eta \in X_1$. Thus, $\mathcal{F}(\lambda,u) = 0$ is equivalent to the following system

$$P\mathcal{F}(\lambda, u_* + \sigma + \eta) = 0, \ \ (I-P)\mathcal{F}(\lambda, u_* + \sigma + \eta) = 0. \tag{4.27}$$

By a simple calculation, we have $(I-P)\mathcal{F}(0,u_*) = 0$ and $(I-P)D_\sigma\mathcal{F}(0,u_*) = D_u\mathcal{F}(0,u_*)$. From the implicit function theorem, there exist an open neighborhood $\mathcal{U}$ of $(0,0)$ in $\mathbb{R}^2$, and a continuously differentiable map $\tilde{\eta} : \mathcal{U} \to X_1$ such that $\tilde{\eta}(0,0) = 0$ and

$$(I-P)\mathcal{F}(\lambda, u_* + \sigma + \tilde{\eta}(\lambda,\sigma)) \equiv 0. \tag{4.28}$$

Substituting $\eta = \tilde{\eta}(\lambda,\sigma)$ into the first equation of (4.27) and then calculating the inner product with l, we obtain that

$$\begin{aligned} G(\lambda,\sigma) &\triangleq \langle l, P\mathcal{F}(\lambda, u_* + \sigma + \tilde{\eta}(\lambda,\sigma))\rangle \\ &= \langle l, \mathcal{F}(\lambda, u_* + \sigma + \tilde{\eta}(\lambda,\sigma))\rangle = 0. \end{aligned} \tag{4.29}$$

From above argument, we reduce the original bifurcation problem to the problem of finding zeros of $G(\lambda,\sigma)$. By (4.28), we see that $(I-P)D_u\mathcal{F}(0,u_*)[1 + \tilde{\eta}_\sigma(0,0)] = 0$. Since $D_u\mathcal{F}(0,u_*)[1] = 0$, $\tilde{\eta}_\sigma \in X_1$ and $D_u\mathcal{F}(0,u_*)_{X_1}$ is an isomorphism, then $\tilde{\eta}_\sigma(0,0) = 0$. Clearly, $G(0,0) = G_\sigma(0,0) = 0$. Notice that

$$G_\lambda(0,0) = u_* \int_\Omega e^{\alpha P(x)} f(x,u_*)\mathrm{d}x + u_* \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,u_*)\mathrm{d}S.$$

By the implicit function theorem, if $G_\lambda(0,0) \neq 0$, then there exist a constant $\delta > 0$ and a continuously differentiable mapping $\sigma \mapsto \lambda_\sigma$ from $(-\delta,\delta) \to \mathbb{R}$ such that $G(\lambda_\sigma,\sigma) \equiv 0$ for $\sigma \in (-\delta,\delta)$, i.e., (1.6) with $\lambda = \lambda_\sigma$ has a positive solution $u_* + \sigma + \tilde{\eta}(\lambda_\sigma,\sigma)$. This combined with Lemma 4.2 implies that $(0,u_*) \in \mathbb{R}\times X$ is not a bifurcation point of the map $\mathcal{F}$ when $G_\lambda(0,0) \neq 0$. Moreover, we have $\mathcal{F}_\lambda(0,u_*) \notin R(D_u\mathcal{F}(0,\cdot))$. Then the following conclusion holds true.

Theorem 4.5. *Assume that f, β satisfy* **(H1)** *with $f \in C^2(\Omega\times\mathbb{R})$, $g \in C^2(\partial\Omega\times\mathbb{R})$, and also satisfy*

(H2)′ $\displaystyle\int_\Omega e^{\alpha P(x)} f(x,u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,u_*)\mathrm{d}S \neq 0$ *for some* $u_* > 0$.

Then there exists no positive solution of (1.6) *bifurcating from* $(0,u_*)$*, that is to say, the solution set of* (1.6) *near* $(0,u_*)$ *consists precisely of the trivial curve* Γ_{u_1}.

Next, we consider the bifurcation problem when $G_\lambda(0,0) = 0$ holds. Under the assumption **(H2)**, we have $\nabla G(0,0) = (0,0)$. To find the zeros of $G(\lambda,\sigma)$, we need to calculate the Hessian matrix of G at $(0,0)$, which has the form:

$$\mathrm{Hess}(G) = \begin{pmatrix} G_{\lambda\lambda}(0,0) & G_{\lambda\sigma}(0,0) \\ G_{\lambda\sigma}(0,0) & G_{\sigma\sigma}(0,0) \end{pmatrix}. \tag{4.30}$$

From Lemma 2.5 in [39], we have the following results:

(i) If $\det \mathrm{Hess}(G) > 0$, then $(0,0)$ is the unique zero of G near $(0,0)$;
(ii) If $\det \mathrm{Hess}(G) < 0$, then there exist two C^{p-1} curves $(\lambda_1(s),\sigma_1(s))$ and $(\lambda_2(s),\sigma_2(s))$ for $s \in (-\delta,\delta)$, such that the solution set of $G(\lambda,\sigma) = 0$ consists of precisely these two curves near $(0,0)$, which satisfy $(\lambda_1(0),\sigma_1(0)) = (\lambda_2(0),\sigma_2(0)) = (0,0)$. Moreover, s can be rescaled so that $(\lambda_1'(0),\sigma_1'(0))$ and $(\lambda_2'(0),\sigma_2'(0))$ are the two linear independent solutions of

$$G_{\lambda\lambda}(0,0)x^2 + 2G_{\lambda\sigma}(0,0)xy + G_{\sigma\sigma}(0,0)y^2 = 0.$$

A straightforward calculation gives that $G_{\sigma\sigma}(0,0) = 0$ and

$$G_{\lambda\sigma}(0,0) = \int_\Omega e^{\alpha P(x)}\big[f(x,u_*) + f_u(x,u_*)u_*\big]\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\big[\beta(x,u_*) + \beta_u(x,u_*)u_*\big]\mathrm{d}S.$$

Notice that $(I-P)\big(D_\lambda\mathcal{F}(0,u_*) + D_u\mathcal{F}(0,u_*)[\tilde{\eta}_\lambda(0,0)]\big) = 0$. This together with the fact that $\tilde{\eta}_\lambda(0,0) \in X_1$ and $D_\lambda\mathcal{F}(0,u_*) \in R(D_u\mathcal{F}(0,u_*))$ means that $D_\lambda\mathcal{F}(0,u_*) + D_u\mathcal{F}(0,u_*)[\tilde{\eta}_\lambda(0,0)] = 0$. Moreover, since

$$\begin{aligned} G_{\lambda\lambda} &= \langle l, D_{\lambda\lambda}\mathcal{F} + 2D_{\lambda u}\mathcal{F}[\tilde{\eta}_\lambda] + D_{uu}\mathcal{F}[\tilde{\eta}_\lambda, \tilde{\eta}_\lambda] + D_u\mathcal{F}[\tilde{\eta}_{\lambda\lambda}]\rangle \\ &= \langle l, D_{\lambda\lambda}\mathcal{F} + 2D_{\lambda u}\mathcal{F}[\tilde{\eta}_\lambda] + D_{uu}\mathcal{F}[\tilde{\eta}_\lambda, \tilde{\eta}_\lambda]\rangle, \end{aligned}$$

we have

$$\begin{aligned} G_{\lambda\lambda}(0,0) = 2\int_\Omega e^{\alpha P(x)}\big[f(x,u_*) + u_* f_u(x,u_*)\big]v_1 \mathrm{d}x \\ + 2\int_{\partial\Omega} e^{\alpha P(x)}\big[\beta(x,u_*) + u_*\beta_u(x,u_*)\big]v_1 \mathrm{d}S, \end{aligned}$$

where v_1 is the unique solution of

$$\begin{cases} \nabla\cdot\big[e^{\alpha P(x)}\nabla v\big] + e^{\alpha P(x)} f(x,u_*)u_* = 0, & x\in\Omega, \\ \partial_{\vec{n}} v = \beta(x,u_*)u_*, & x\in\partial\Omega, \\ \int_\Omega v(x)\mathrm{d}x = 0. \end{cases} \tag{4.31}$$

Thus, we have the following result.

Theorem 4.6. *Assume that f, β satisfy* **(H1)** *with $f \in C^2(\Omega\times\mathbb{R})$, $g \in C^2(\partial\Omega\times\mathbb{R})$,* **(H2)** *and*

(H4) $\int_\Omega e^{\alpha P(x)} f_u(x,u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\beta_u(x,u_*)\mathrm{d}S \neq 0$ *for some $u_* > 0$.*

Then the solution set of (1.6) near $(0,u_*)$ consists exactly of the trivial solution curve Γ_{u_1} and the curve

$$\Sigma_1 = \{(\lambda_1(s), u_1(s)) : s\in(-\delta,\delta)\subset\mathbb{R}\},$$

where $\lambda_1(s)$ and $u_1(s) = u_* + \sigma_1(s) + \tilde{\eta}(\lambda_1(s),\sigma_1(s))$ are C^1 functions such that $\lambda_1(0) = \sigma_1(0) = 0$, $\lambda_1'(0) = 1$ and

$$\sigma_1'(0) = -\frac{\int_\Omega e^{\alpha P(x)}\big[f(x,u_*) + u_* f_u(x,u_*)\big]v_1\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\big[\beta(x,u_*) + u_*\beta_u(x,u_*)\big]v_1\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}\big[f(x,u_*) + f_u(x,u_*)u_*\big]\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\big[\beta(x,u_*) + \beta_u(x,u_*)u_*\big]\mathrm{d}S},$$

with v_1 defined as in (4.31).

Proof. If the f, β satisfy **(H1)**, **(H2)** and **(H4)**, then $\nabla G(0,0) = (0,0)$ and $\det \mathrm{Hess}(G) < 0$. Thus, there exist two C^{p-1} curves $(\lambda_1(s),\sigma_1(s))$ and $(\lambda_2(s),\sigma_2(s))$ for $s\in(-\delta,\delta)$, which satisfy $(\lambda_1(0),\sigma_1(0)) = (\lambda_2(0),\sigma_2(0)) = (0,0)$, such that the solution set of $G(\lambda,\sigma) = 0$ consists of precisely these two curves near $(0,0)$ and that $(\lambda_1'(0),\sigma_1'(0)) = (1,\sigma_1'(0))$ and $(\lambda_2'(0),\sigma_2'(0)) = (0,1)$. It is noticed that the solution curve $(\lambda_2(s),\sigma_2(s)) = (0,1)s + o(s)$ of $G = 0$ is corresponding to the trivial branch $\Gamma_{u_1} = \{(0,u_1) : u_1 > 0\}$. Hence, we obtain the existence of two solution curves of (1.6), which are tangent to each other at the bifurcation point. □

In what follows, we study the stability of bifurcating solution $(\lambda_1(s), u_1(s))$ in Theorem 4.6. Consider the linear eigenvalue problem (3.1) at $(\lambda_1(s), u_1(s))$, which has the form:

$$\begin{cases} -e^{-\alpha P(x)}\nabla\cdot\big[e^{\alpha P(x)}\nabla\psi\big]-\lambda_1(s)\big[f(x,u_1(s))+u_1(s)f_u(x,u_1(s))\big]\psi=\mu\psi, & x\in\Omega,\\ \partial_{\vec{n}}\psi-\lambda_1(s)\big[\beta(x,u_1(s))+u_1(s)\beta_u(x,u_1(s))\big]\psi=\mu\psi, & x\in\partial\Omega. \end{cases} \tag{4.32}$$

We still denote by $(\mu_1(s), \psi_1(s))$ the principal eigen-pair of problem (4.32). Obviously, $\mu_1(0) = 0$ and $\psi_1(0) = 1$. Multiplying the first equation of (4.32) by $e^{\alpha P(x)}$ and then integrating over Ω, it follows that

$$\begin{aligned} &\mu_1(s)\Big(\int_\Omega e^{\alpha P(x)}\psi_1(s)\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\psi_1(s)\mathrm{d}S\Big)\\ =&-\lambda_1(s)\Big(\int_\Omega e^{\alpha P(x)}\big[f(x,u_1)+f_u(x,u_1)u_1\big]\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\big[\beta(x,u_1)+\beta_u(x,u_1)u_1\big]\mathrm{d}S\Big). \end{aligned}$$

Taking $s \to 0$, we see that

$$\lim_{s\to 0}\frac{\mu_1(s)}{\lambda_1(s)}=-\frac{G_{\lambda\sigma}(0,0)}{\int_\Omega e^{\alpha P(x)}\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\mathrm{d}S}.$$

Therefore, we can conclude that under the assumptions **(H1)**, **(H2)** and **(H4)**, there exists a positive solution curve $\Sigma_1 = \{(\mu_1(s), u_1(s)) : s \in (0, \delta) \in \mathbb{R}, \delta > 0\}$ of (1.6) bifurcating from the trivial branch $\Gamma_{u_1} = \{(0, u_1) : u_1 > 0\}$. Moreover, when $G_{\lambda\sigma}(0, 0) < 0$, the positive bifurcating solution of (1.6) is stable (resp. unstable) for $s \in (0, \delta)$ (resp. $s \in (-\delta, 0)$); when $G_{\lambda\sigma}(0, 0) > 0$, it becomes unstable (resp. stable) for $s \in (0, \delta)$ (resp. $s \in (-\delta, 0)$).

4.3. Local bifurcation from $(0, 0)$

It is easy to see that in the previous subsection, if u_* is replaced by 0, then there must be $G_\lambda(0, 0) = 0$, and the argument for Theorem 4.6 can also be applied to the point $(\lambda, u) = (0, 0)$ with some modification. Thus we have the following conclusion.

Theorem 4.7. *Assume that* f, β *satisfy* **(H1)** *with* $f \in C^2(\Omega\times\mathbb{R})$, $g \in C^2(\partial\Omega\times\mathbb{R})$, *and the condition*

(H3)′ $$\int_\Omega e^{\alpha P(x)}f(x,0)\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\mathrm{d}S\neq 0.$$

Then the nonnegative solution set of (1.6) *near* $(0, 0)$ *is the union of* Γ_0 *and* Γ_{u_1}.

In fact, Theorem 4.7 is valid since det Hess$(G) < 0$ at the bifurcation point $(0, 0)$, where $G(\lambda, \sigma) = \langle l, \mathcal{F}(\lambda, \sigma + \tilde{\eta}(\lambda, \sigma))\rangle$. However, when the condition

(H3)″ $$\int_\Omega e^{\alpha P(x)}f(x,0)\mathrm{d}x+\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\mathrm{d}S=0$$

is satisfied, we have $\det \mathrm{Hess}(G) = 0$ at the bifurcation point $(0, 0)$, and Lemma 2.5 of Liu, Shi and Wang [39] can not be applied to G. In this degenerate case, we need to further investigate the bifurcation problem at $(0, 0)$. Assume that $f \in C^3(\Omega \times \mathbb{R})$, $g \in C^3(\partial\Omega \times \mathbb{R})$. Then $\tilde{\eta}$ and G are C^3 in a neighborhood $\mathcal{U}$ of $(\lambda, \sigma) = (0, 0)$. Since $\mathcal{F}(\lambda, 0) \equiv 0$, it follows that $\tilde{\eta}(\lambda, 0) \equiv 0$, hence $\tilde{\eta}_\lambda(0, 0) = 0$ and $\tilde{\eta}_{\lambda\lambda}(0, 0) = 0$. Define $\mathscr{F}(\lambda, \sigma) := (I - P)\mathcal{F}(\lambda, \sigma + \tilde{\eta}(\lambda, \sigma)) \equiv 0$ for $(\lambda, \sigma) \in \mathcal{U}$. It is easy to calculate that

$$\frac{\partial \mathscr{F}}{\partial \sigma} = (I - P)D_u\mathcal{F}(0, 0)[1 + \tilde{\eta}_\sigma(0, 0)] = D_u\mathcal{F}(0, 0)[\tilde{\eta}_\sigma(0, 0)] = 0,$$

thus $\tilde{\eta}_\sigma(0, 0) = 0$ from $D_u\mathcal{F}(0, 0)[1] = 0$, $D_u\mathcal{F}(0, 0)|_{X_1} : X_1 \to R(D_u\mathcal{F}(0, 0))$ is an isomorphism and $\tilde{\eta}_\sigma(0, 0) \in X_1$. For the second derivatives of $\mathscr{F}$, we calculate that

$$\begin{aligned}\frac{\partial^2 \mathscr{F}}{\partial\lambda\partial\sigma}(0, 0) &= (I - P)\big(D_{\lambda u}\mathcal{F}(0, 0)\big[1 + \tilde{\eta}_\sigma(0, 0)\big] + D_{uu}\mathcal{F}(0, 0)\big[1 + \tilde{\eta}_\sigma(0, 0), \tilde{\eta}_\lambda(0, 0)\big] \\ &\quad + D_u\mathcal{F}(0, 0)\big[\tilde{\eta}_{\lambda\sigma}(0, 0)\big]\big) \\ &= D_{\lambda u}\mathcal{F}(0, 0)\big[1\big] + D_u\mathcal{F}(0, 0)\big[\tilde{\eta}_{\lambda\sigma}(0, 0)\big] = 0.\end{aligned}$$

In view of **(H3)**$''$, there holds that $D_{\lambda u}\mathcal{F}(0, 0)\big[1\big] \in R(D_u\mathcal{F}(0, 0))$, and then $\tilde{\eta}_{\lambda\sigma}(0, 0) = \xi_1$ is the unique solution of

$$\begin{cases} \nabla \cdot \big[e^{\alpha P(x)}\nabla\xi\big] + e^{\alpha P(x)} f(x, 0) = 0, & x \in \Omega, \\ \partial_{\vec{n}}\xi = \beta(x, 0), & x \in \partial\Omega, \\ \int_\Omega \xi(x)\mathrm{d}x = 0. \end{cases} \tag{4.33}$$

Similarly, the equation

$$\begin{aligned}\frac{\partial^2 \mathscr{F}}{\partial\sigma^2}(0, 0) &= (I - P)\left(D_{uu}\mathcal{F}(0, 0)\big[1 + \tilde{\eta}_\sigma(0, 0)\big]^2 + D_u\mathcal{F}(0, 0)\big[\tilde{\eta}_{\sigma\sigma}(0, 0)\big]\right) \\ &= D_{uu}\mathcal{F}(0, 0)\big[1\big]^2 + D_u\mathcal{F}(0, 0)\big[\tilde{\eta}_{\sigma\sigma}(0, 0)\big] = 0\end{aligned}$$

implies that $\tilde{\eta}_{\sigma\sigma}(0, 0) = 0$ since $D_{uu}\mathcal{F}(0, 0)\big[1\big]^2 = 0$ and $\tilde{\eta}_{\sigma\sigma}(0, 0) \in X_1$.

Since $\tilde{\eta}(\lambda, 0) \equiv 0$, there should be $G(\lambda, 0) \equiv 0$. Now, we define

$$H(\lambda, \sigma) = \begin{cases} \dfrac{1}{\sigma} G(\lambda, \sigma), & \text{if } \sigma \neq 0, \\ G_\sigma(\lambda, 0), & \text{if } \sigma = 0. \end{cases} \tag{4.34}$$

Firstly, we check that $H(\lambda, \sigma)$ is C^2 at $\sigma = 0$ in $\mathcal{U}$. We calculate that

$$\begin{aligned}H_\sigma(\lambda, 0) &= \lim_{\sigma\to 0}\frac{1}{\sigma}(H(\lambda, \sigma) - H(\lambda, 0)) = \lim_{\sigma\to 0}\frac{1}{\sigma}\left(\frac{1}{\sigma}G(\lambda, \sigma) - G_\sigma(\lambda, 0)\right) \\ &= \lim_{\sigma\to 0}\frac{1}{\sigma^2}\big(G(\lambda, \sigma) - G(\lambda, 0) - G_\sigma(\lambda, 0)\sigma\big) = \frac{1}{2}G_{\sigma\sigma}(\lambda, 0),\end{aligned}$$

$$\begin{aligned}
H_{\lambda\sigma}(\lambda,0) &= \lim_{\sigma\to 0}\frac{1}{\sigma}(H_\lambda(\lambda,\sigma)-H_\lambda(\lambda,0)) = \lim_{\sigma\to 0}\frac{1}{\sigma}\left(\frac{1}{\sigma}G_\lambda(\lambda,\sigma)-G_{\lambda\sigma}(\lambda,0)\right)\\
&= \lim_{\sigma\to 0}\frac{1}{\sigma^2}\big(G_\lambda(\lambda,\sigma)-G_\lambda(\lambda,0)-G_{\lambda\sigma}(\lambda,0)\sigma\big) = \frac{1}{2}G_{\lambda\sigma\sigma}(\lambda,0),\\
H_\lambda(\lambda,0) &= G_{\lambda\sigma}(\lambda,0), \qquad H_{\lambda\lambda}(\lambda,0) = G_{\lambda\lambda\sigma}(\lambda,0),
\end{aligned}$$

and

$$\begin{aligned}
H_{\sigma\sigma}(\lambda,0) &= \lim_{\sigma\to 0}\frac{1}{\sigma}(H_\sigma(\lambda,\sigma)-H_\sigma(\lambda,0))\\
&= \lim_{\sigma\to 0}\frac{1}{\sigma}\left(\frac{-G(\lambda,\sigma)+\sigma G_\sigma(\lambda,\sigma)}{\sigma^2}-\frac{1}{2}G_{\sigma\sigma}(\lambda,0)\right)\\
&= \lim_{\sigma\to 0}\frac{1}{\sigma^3}\left(-G(\lambda,\sigma)+\sigma G_\sigma(\lambda,\sigma)-\frac{1}{2}G_{\sigma\sigma}(\lambda,0)\right) = \frac{1}{3}G_{\sigma\sigma\sigma}(\lambda,0).
\end{aligned}$$

Thus H_σ and $H_{\sigma\sigma}$ are well-defined for $\sigma = 0$. In addition,

$$\begin{aligned}
H_{\sigma\sigma}(\lambda,\sigma)-H_{\sigma\sigma}(\lambda,0) &= \frac{2}{\sigma^3}\left[G(\lambda,\sigma)-\sigma G_\sigma(\lambda,\sigma)+\frac{\sigma^2}{2}G_{\sigma\sigma}(\lambda,\sigma)-\frac{\sigma^3}{6}G_{\sigma\sigma\sigma}(\lambda,0)\right]\\
&= o(\sigma),\\
H_{\lambda\lambda}(\lambda,\sigma)-H_{\lambda\lambda}(\lambda,0) &= \frac{1}{\sigma}G_{\lambda\lambda}(\lambda,\sigma)-G_{\lambda\lambda\sigma}(\lambda,0)\\
&= \frac{1}{\sigma}\big[G_{\lambda\lambda}(\lambda,\sigma)-G_{\lambda\lambda}(\lambda,0)-G_{\lambda\lambda\sigma}(\lambda,0)\sigma\big] = o(\sigma)
\end{aligned}$$

and

$$\begin{aligned}
H_{\lambda\sigma}(\lambda,\sigma)-H_{\lambda\sigma}(\lambda,0) &= -\frac{1}{\sigma^2}G_\lambda(\lambda,\sigma)+\frac{1}{\sigma}G_{\lambda\sigma}(\lambda,\sigma)-\frac{1}{2}G_{\lambda\sigma\sigma}(\lambda,0)\\
&= -\frac{1}{\sigma^2}\left[G_\lambda(\lambda,\sigma)-\sigma G_{\lambda\sigma}(\lambda,\sigma)+\frac{1}{2}G_{\lambda\sigma\sigma}(\lambda,0)\sigma^2\right] = o(\sigma),
\end{aligned}$$

which means that $H \in C^2$ at $\sigma = 0$ in $\mathcal{U}$.

Next, we show that $H(0,0) = 0$, $\nabla H(0,0) = (H_\lambda(0,0), H_\sigma(0,0)) = (0,0)$, and the Hessian matrix $\mathrm{Hess}(H)$ is non-degenerate. Calculating the partial derivatives of H at $(0,0)$, it follows that

$$\begin{aligned}
H(0,0) &= G_\sigma(0,0) = \big\langle l, D_u\mathcal{F}(0,0)\big[1+\tilde{\eta}_\sigma(0,0)\big]\big\rangle = 0,\\
H_\lambda(0,0) &= G_{\lambda\sigma}(0,0)\\
&= \big\langle l, D_{\lambda u}\mathcal{F}(0,0)\big[1+\tilde{\eta}_\sigma(0,0)\big] + D_{uu}\mathcal{F}(0,0)\big[\tilde{\eta}_\lambda(0,0), 1+\tilde{\eta}_\sigma(0,0)\big]\\
&\quad + D_u\mathcal{F}(0,0)\big[\tilde{\eta}_{\lambda\sigma}(0,0)\big]\big\rangle\\
&= \big\langle l, D_{\lambda u}\mathcal{F}(0,0)\big[1\big]\big\rangle = \int_\Omega e^{\alpha P(x)} f(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\mathrm{d}S = 0,
\end{aligned}$$

and

$$\begin{aligned} H_\sigma(0,0) &= \frac{1}{2} G_{\sigma\sigma}(\lambda, 0) \\ &= \frac{1}{2} \Big\langle l, D_{uu}\mathcal{F}(0,0)\big[1+\tilde{\eta}_\sigma(0,0)\big]^2 + D_u\mathcal{F}(0,0)\big[\tilde{\eta}_{\sigma\sigma}(0,0)\big]\Big\rangle \\ &= \frac{1}{2} \Big\langle l, D_{uu}\mathcal{F}(0,0)\big[1\big]^2\Big\rangle = 0. \end{aligned}$$

Recall that $\tilde{\eta}_\sigma(0,0) = \tilde{\eta}_\lambda(0,0) = \tilde{\eta}_{\lambda\lambda}(0,0) = \tilde{\eta}_{\sigma\sigma}(0,0) = 0$, and $\tilde{\eta}_{\lambda\sigma}(0,0) = \xi_1$ is the unique solution of (4.33). For the Hessian matrix

$$\mathrm{Hess}(H) = \begin{pmatrix} H_{\lambda\lambda} & H_{\lambda\sigma} \\ H_{\lambda\sigma} & H_{\sigma\sigma} \end{pmatrix},$$

we have the following evaluations at (0, 0):

$$\begin{aligned} H_{\lambda\lambda}(0,0) &= G_{\lambda\lambda\sigma}(0,0) \\ &= \Big\langle l, D_{\lambda\lambda u}\mathcal{F}\big[1+\tilde{\eta}_\sigma\big] + 2D_{\lambda uu}\mathcal{F}\big[\tilde{\eta}_\lambda, 1+\tilde{\eta}_\lambda\big] + 2D_{\lambda u}\mathcal{F}\big[\tilde{\eta}_{\lambda\sigma}\big] + 2D_{uu}\mathcal{F}\big[\tilde{\eta}_{\lambda\sigma}, \tilde{\eta}_\lambda\big] \\ &\quad + D_u\mathcal{F}\big[\tilde{\eta}_{\lambda\lambda\sigma}\big] + D_{uuu}\mathcal{F}\big[\tilde{\eta}_\lambda, \tilde{\eta}_\lambda, 1+\tilde{\eta}_\sigma\big] + D_{uu}\mathcal{F}\big[\tilde{\eta}_{\lambda\lambda}, 1+\tilde{\eta}_\sigma\big]\Big\rangle \\ &= \Big\langle l, D_{\lambda\lambda u}\mathcal{F}(0,0)\big[1\big] + 2D_{\lambda u}\mathcal{F}(0,0)\big[\xi_1\big]\Big\rangle \\ &= 2\int_\Omega e^{\alpha P(x)} f(x,0)\xi_1(x)\mathrm{d}x + 2\int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\xi_1(x)\mathrm{d}S = 2\int_\Omega e^{\alpha P(x)}|\nabla\xi_1|^2\mathrm{d}x, \\ H_{\lambda\sigma}(0,0) &= \tfrac{1}{2} G_{\lambda\sigma\sigma}(0,0) \\ &= \tfrac{1}{2}\Big\langle l, D_{\lambda uu}\mathcal{F}\big[1+\tilde{\eta}_\sigma\big]^2 + D_{\lambda u}\mathcal{F}\big[\tilde{\eta}_{\sigma\sigma}\big] + 2D_{uu}\mathcal{F}\big[\tilde{\eta}_{\lambda\sigma}, 1+\tilde{\eta}_\sigma\big] + D_u\mathcal{F}\big[\tilde{\eta}_{\lambda\sigma\sigma}\big] \\ &\quad + D_{uuu}\mathcal{F}\big[\tilde{\eta}_\lambda, 1+\tilde{\eta}_\sigma, 1+\tilde{\eta}_\sigma\big] + D_{uu}\mathcal{F}\big[\tilde{\eta}_\lambda, \tilde{\eta}_{\sigma\sigma}\big]\Big\rangle \\ &= \tfrac{1}{2}\Big\langle l, D_{\lambda uu}\mathcal{F}(0,0)\big[1\big]^2 + 2D_{uu}\mathcal{F}(0,0)\big[\tilde{\eta}_{\lambda\sigma}, 1\big]\Big\rangle \\ &= \int_\Omega e^{\alpha P(x)} f_u(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\beta_u(x,0)\mathrm{d}S, \end{aligned}$$

and

$$\begin{aligned} H_{\sigma\sigma}(0,0) &= \frac{1}{3} G_{\sigma\sigma\sigma}(0,0) \\ &= \frac{1}{3}\Big\langle l, D_{uuu}\mathcal{F}\big[1+\tilde{\eta}_\sigma\big]^3 + 3D_{uu}\mathcal{F}\big[\tilde{\eta}_{\sigma\sigma}, 1+\tilde{\eta}_\sigma\big] + D_u\mathcal{F}\big[\tilde{\eta}_{\sigma\sigma\sigma}\big]\Big\rangle \\ &= \frac{1}{3}\Big\langle l, D_{uuu}\mathcal{F}(0,0)\big[1\big]^3\Big\rangle = 0. \end{aligned}$$

Hence from Lemma 2.5 of Liu, Shi and Wang [39], we see that if $\det \mathrm{Hess}(H) < 0$, then the solution set of $H(\lambda, \sigma) = 0$ near $(\lambda, \sigma) = (0, 0)$ is a pair of intersecting curves. Consequently, the solution set of (1.6) near $(\lambda, u) = (0, 0)$ is precisely the union of trivial solution curve Γ_0

and the pair of intersecting curves which solves $H(\lambda, \sigma) = 0$. Here, the two intersecting curves solving $H(\lambda, \sigma) = 0$ have the form

$$\{(\lambda_i(s), u_i(s)) : |s| < \delta\} \ \ (i = 1, 2)$$

for some $\delta > 0$ with $u_i(s) = \sigma_i(s) + \tilde{\eta}_i(\lambda_i(s), u_i(s))$, where $(\lambda_1'(0), \sigma_1'(0))$ and $(\lambda_2'(0), \sigma_2'(0))$ are non-zero linear independent solutions of

$$H_{\lambda\lambda}(0, 0)x^2 + 2H_{\lambda\sigma}(0, 0)xy = 0.$$

Then we choose $(\lambda_1'(0), \sigma_1'(0)) = (0, 1)$, $(\lambda_2'(0), \sigma_2'(0)) = (1, -\frac{H_{\lambda\lambda}(0,0)}{2H_{\lambda\sigma}(0,0)})$ such that the curve $\{(\lambda_1(s), u_1(s))\}$ is consistent with the trivial solution set Γ_{u_1}, while the curve $\Sigma_2 = \{(\lambda_2(s), u_2(s))\}$ must be distinct from Γ_0 or Γ_{u_1}. Thus, the following theorem is established.

Theorem 4.8. *Assume that f, β satisfy* **(H1)**, **(H3)**$''$ *with $f \in C^3(\Omega \times \mathbb{R})$, $g \in C^3(\partial\Omega \times \mathbb{R})$, and the condition*

(H5) $$\int_\Omega e^{\alpha P(x)} f_u(x, 0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta_u(x, 0)\mathrm{d}S \neq 0.$$

Then the solution set of (1.6) *near* $(0, 0)$ *consists exactly of the curves* Γ_0, Γ_{u_1} *and*

$$\Sigma_2 = \{(\lambda_2(s), u_2(s)) : s \in (-\delta, \delta) \subset \mathbb{R}\},$$

where $\lambda_2(s)$ and $u_2(s) = \sigma_2(s) + \tilde{\eta}(\lambda_2(s), \sigma_2(s))$ are C^1 functions such that $\lambda_2(0) = \sigma_2(0) = 0$, $\lambda_2'(0) = 1$ and

$$\sigma_2'(0) = -\frac{\int_\Omega e^{\alpha P(x)} |\nabla \xi_1|^2 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} f_u(x, 0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta_u(x, 0)\mathrm{d}S},$$

with ξ_1 defined as in (4.33).

Remark 4.9. From Theorem 4.8, it can be seen that if $\int_\Omega e^{\alpha P(x)} f_u(x, 0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta_u(x, 0)\mathrm{d}S < 0$, then there exists a negative solution curve Σ_2^- of (1.6) bifurcating from the line of trivial solutions Γ_0 and Γ_{u_1}, which has the form $\Sigma_2^- = \{(\lambda_2(s), u_2(s)) : s \in I = (-\delta, 0) \subset \mathbb{R}\}$. However, this solution has no biological significance.

Finally, we investigate the stability of positive bifurcating solution $(\lambda_2(s), u_2(s))$ for $s \in (0, \delta)$ in Theorem 4.8. Consider the linear eigenvalue problem (3.1) at $(\lambda_2(s), u_2(s))$, which has the form:

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi\right] - \lambda_2(s)\left[f(x, u_2(s)) + u_2(s) f_u(x, u_2(s))\right]\psi = \mu\psi, & x \in \Omega, \\ \partial_{\vec{n}} \psi - \lambda_2(s)\left[\beta(x, u_2(s)) + u_2(s)\beta_u(x, u_2(s))\right]\psi = \mu\psi, & x \in \partial\Omega. \end{cases} \tag{4.35}$$

We still denote by $(\mu_1(s), \psi_1(s))$ the principal eigen-pair of problem (4.35). Obviously, $\mu_1(0) = 0$ and $\psi_1(0) = 1$. Moreover, $\lambda_2(0) = u_2(0) = 0$, $\lambda_2'(0) = 1$ and $u_2'(0) = \sigma_2'(0)$. Substituting $\lambda =$

$\lambda_2(s), u = u_2(s)$ into (1.6), differentiating twice with respect to s, and then letting $s = 0$, we obtain that

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla u_2''(0)\right] = 2e^{\alpha P(x)} f(x,0)\sigma_2'(0), & x \in \Omega, \\ \partial_{\vec{n}} u_2''(0) = 2\beta(x,0)\sigma_2'(0), & x \in \partial\Omega, \end{cases}$$

which leads to that $u_2''(0) = 2\sigma_2'(0)\xi_1$, where ξ_1 is the unique solution of (4.33). Similarly by differentiating (4.35) and setting $s = 0$, we get

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi_1'(0)\right] - f(x,0) = \mu_1'(0), & x \in \Omega, \\ \partial_{\vec{n}} \psi_1'(0) - \beta(x,0) = \mu_1'(0), & x \in \partial\Omega. \end{cases} \tag{4.36}$$

By multiplying the first equation of (4.36) by $e^{\alpha P(x)}$ and integrating over Ω, it is inferred that $\mu_1'(0) = 0$ and then $\psi_1'(0) = \xi_1$. Next, differentiating (4.35) twice by s and taking $s = 0$, we have

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi_1''(0)\right] & \\ \quad -2f(x,0)\xi_1 - 4f_u(x,0)\sigma_2'(0) - \lambda_2''(0) f(x,0) = \mu_1''(0), & x \in \Omega, \\ \partial_{\vec{n}} \psi_1''(0) - 2\beta(x,0)\xi_1 - 4\beta_u(x,0)\sigma_2'(0) - \lambda_2''(0)\beta(x,0) = \mu_1''(0), & x \in \partial\Omega, \end{cases} \tag{4.37}$$

where we have used the fact that $\mu_1'(0) = 0$ and $\psi_1'(0) = \xi_1$. By the assumption **(H3)**$''$, multiplying the first equation of (4.37) by $e^{\alpha P(x)}$ and integrating over Ω, we derive that

$$\begin{aligned} &\mu_1''(0)\Big(\int_\Omega e^{\alpha P(x)} \mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \mathrm{d}S\Big) \\ &\quad = -2\Big(\int_\Omega e^{\alpha P(x)} f(x,0)\xi_1(x)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta(x,0)\xi_1(x)\mathrm{d}S\Big) \\ &\qquad - 4\sigma_2'(0)\Big(\int_\Omega e^{\alpha P(x)} f_u(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)} \beta_u(x,0)\mathrm{d}S\Big) \\ &\quad = 2\int_\Omega e^{\alpha P(x)} |\nabla \xi_1|^2 \mathrm{d}x > 0, \end{aligned} \tag{4.38}$$

which implies that $\mu_1''(0) > 0$.

Therefore, we can conclude from the above arguments and Theorem 3.4 that under the assumptions **(H1)**, **(H3$''$)** and **(H5)**, the trivial solution $(\lambda, 0)$ of (1.6) is unstable for $\lambda \in (0, \infty)$ and the positive bifurcating solution $(\lambda_2(s), u_2(s))$ is stable for $s \in (0, \delta)$.

5. Application to a parabolic equation with monostable nonlinear boundary condition

In this section, we consider a special case of (1.5) with $f(x,u) \equiv 0$ and $\beta(x,u) = r(x)b(u)$, i.e.,

$$\begin{cases} u_t = e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla u \right], & x \in \Omega, t > 0, \\ \partial_{\vec{n}} u = \lambda r(x) b(u) u, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \Omega, \end{cases} \tag{5.1}$$

where $\Omega \in \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive parameter, $r : \partial\Omega \to \mathbb{R}$ is of class $C^{1+\theta}(\partial\Omega)$ for some $\theta \in (0, 1)$ and $b : \mathbb{R} \to \mathbb{R}$ is C^3 function satisfying

(A1) $\begin{cases} b(0) > 0, \; b(1) = 0, \; b'(1) < 0, \; [ub(u)]'' < 0 \text{ in } (0,1), \\ b > 0 \text{ in } (0,1), \; b < 0 \text{ in } (1, +\infty). \end{cases}$

Clearly, $b(u) = 1 - u$ satisfies the condition **(A1)**. Moreover, for the parabolic problem above we have

(A2) $\mathfrak{X} := \{u \in H^1(\Omega) : 0 \leq u(x) \leq 1 \text{ a.e. } x \in \Omega\}$ is the phase space for (5.1)

keeping analogy with several problems occurring in population genetics where solutions with $0 \leq u \leq 1$ are of interest.

Consider the steady state solutions of (5.1) satisfying $0 \leq u \leq 1$, which satisfy

$$\begin{cases} \nabla \cdot \left[e^{\alpha P(x)} \nabla u \right] = 0, & x \in \Omega, \\ \partial_{\vec{n}} u = \lambda r(x) b(u) u, & x \in \partial\Omega. \end{cases} \tag{5.2}$$

For application of the results of Section 4, we define a nonlinear mapping $\mathcal{F} : \mathbb{R} \times W_l^2(\Omega) \to L^l(\Omega) \times W_l^{1-\frac{1}{l}}(\partial\Omega), l > N$ as

$$\mathcal{F}(\lambda, u) = \left(\nabla \cdot \left[e^{\alpha P(x)} \nabla u \right], \partial_{\vec{n}} u - \lambda r(x) b(u) u \right). \tag{5.3}$$

The aim of this section is to study the bifurcation and stability structures of steady states of (5.1). Assume that b satisfies **(A1)**. Clearly, (5.2) has the trivial solution curves

$$\Gamma_0 := \{(\lambda, 0) : \lambda > 0\}, \quad \Gamma_1 := \{(\lambda, 1) : \lambda > 0\}$$

and

$$\Gamma_{u_1} := \{(0, u_1) : 0 < u_1 < 1\}.$$

We first investigate bifurcation from the solution set Γ_0 and Γ_1 as follows:

Proposition 5.1. *Suppose that* **(A1)** *holds true and $r : \partial\Omega \to \mathbb{R}$ is sign-changing on $\partial\Omega$.*

(i) If

$$\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S = 0, \tag{5.4}$$

then there is no bifurcation from Γ_0 and Γ_1.

(ii) If

$$\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S < 0, \tag{5.5}$$

then $(\lambda_1, 0)$ is a bifurcation point with respect to the trivial branch Γ_0 and there is no bifurcation from Γ_1, where λ_1 is the unique positive principal eigenvalue of

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla v \right] = 0, & x \in \Omega, \\ \partial_{\vec{n}} v = \lambda r(x) b(0) v, & x \in \partial\Omega. \end{cases} \tag{5.6}$$

(iii) If

$$\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S > 0, \tag{5.7}$$

then $(\tilde{\lambda}_1, 1)$ is a bifurcation point with respect to the trivial branch Γ_1 and there is no bifurcation from Γ_0, where $\tilde{\lambda}_1$ is the unique positive principal eigenvalue of

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \phi \right] = 0, & x \in \Omega, \\ \partial_{\vec{n}} \psi = \lambda r(x) b'(1) \phi, & x \in \partial\Omega. \end{cases}$$

Proof. (i) We only prove the case related to Γ_0. Suppose that $\hat{\lambda} > 0$ is a bifurcation point with respect to Γ_0. From Lemma 4.2 (i), then $\hat{\lambda}$ is the principal eigenvalue of (5.6), which is impossible by (5.4) and Theorem 2.2 (iii). The proof for Γ_1 is similar.

(ii) It can be seen from Theorem 2.2 that when (5.5) holds, (5.6) admits a principal eigenvalue $\lambda_1 > 0$ since $b(0) > 0$. Thus, λ_1 is a bifurcation point with respect to Γ_0. On the other hand, since $b'(1) < 0$ and (5.5) holds, there is no bifurcation from Γ_1. Finally, part (iii) is similar to part (ii). □

In Proposition 5.1, the results of part (ii) and part (iii) seem to be symmetric. Actually, the change of variable $\tilde{u} = 1 - u$ allows us to transform all analysis made for Γ_0 and Γ_1, yielding relatively symmetric results that can be read by interchanging the roles of $u \equiv 0$ and $u \equiv 1$. In the following, we mainly consider the cases (5.4) and (5.5).

5.1. Case (5.4)

It follows from Proposition 5.1 (i) that no bifurcation occurs from Γ_0 and Γ_1. That is, the bifurcation theory for simple eigenvalue can not be applied to Γ_0 and Γ_1. Indeed, we can calculate that

$$D_u \mathcal{F}(0, \bar{u})[\phi] = \left(\nabla \cdot \left[e^{\alpha P(x)} \nabla \phi \right], \frac{\partial \phi}{\partial \vec{n}} \right),$$

$$D_{\lambda u} \mathcal{F}(0, \bar{u})[\phi] = \left(0, -r(x)[b(\bar{u}) + \bar{u} b'(\bar{u})]\phi \right)$$

for $\phi \in W_l^2(\Omega)$, where $\bar{u} = 0$ or $\bar{u} = 1$. Since $N(D_u\mathcal{F}(0, \bar{u})) = \text{span}\{1\}$, by (5.4), it holds that $D_{\lambda u}\mathcal{F}(0, \bar{u})[1] \in R(D_u\mathcal{F}(0, \bar{u}))$. Then the transversality condition of Crandall-Babinowitz theorem [20] is not satisfied. Moreover, we see that the conditions in Theorems 4.6, 4.7 and 4.8 are also not satisfied. So we need to use other approach to study bifurcation from $(0, \bar{u})$ or Γ_{u_1}.

In the following, we perform a Lyapunov-Schmidt reduction to show the solution set bifurcating from Γ_{u_1}. Notice that $D_u\mathcal{F}(0, u_1) = \left(\nabla \cdot \left[e^{\alpha P(x)}\nabla\right], \frac{\partial}{\partial \vec{n}}\right)$ and $N(D_u\mathcal{F}(0, u_1)) = \text{span}\{1\}$ for any constant $u_1 > 0$. Make the decomposition

$$X = N(D_u\mathcal{F}(0, u_1)) \oplus X_1 \ \text{ and } \ Y = R(D_u\mathcal{F}(0, u_1)) \oplus Y_1.$$

Define the operator $l \in Y^*$ by

$$\langle l, y\rangle = \int_\Omega y_1(x)\mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)} y_2(x)\mathrm{d}S, \quad \forall y = (y_1, y_2) \in Y,$$

and then $N(l) = R(D_u\mathcal{F}(0, u_1))$. Now, we let P be a projection operator from Y to Y_1 along $R(D_u\mathcal{F}(0, u_1))$, namely, $Py = \langle \psi, y\rangle\varphi$ for $y \in Y$. Set $u = u_1 + \sigma + \eta$, where $\sigma \in \mathbb{R}$ and $\eta \in X_1$. Following the argument in Section 4.2, there exist an open neighborhood $\mathcal{U}$ of $(0, 0)$ in $\mathbb{R}^2$, and a continuously differentiable map $\tilde{\eta} : \mathcal{U} \to X_1$ such that $\tilde{\eta}(0, 0) = \tilde{\eta}_\sigma(0, 0) = 0$ and

$$(I - P)\mathcal{F}(\lambda, u_1 + \sigma + \tilde{\eta}(\lambda, \sigma)) \equiv 0. \tag{5.8}$$

Then solving (5.2) is equivalent to finding zeros of $G(\lambda, \sigma) = 0$, where

$$\begin{aligned} G(\lambda, \sigma) &\triangleq \langle l, P\mathcal{F}(\lambda, u_1 + \sigma + \tilde{\eta}(\lambda, \sigma))\rangle \\ &= \langle l, \mathcal{F}(\lambda, u_1 + \sigma + \tilde{\eta}(\lambda, \sigma))\rangle \\ &= \lambda \int_{\partial\Omega} e^{\alpha P(x)} r(x) b(u_1 + \sigma + \tilde{\eta}(\lambda, \sigma)) \cdot (u_1 + \sigma + \tilde{\eta}(\lambda, \sigma))\mathrm{d}S. \end{aligned} \tag{5.9}$$

In case (5.4), there holds that $D_\lambda\mathcal{F}(0, u_1) \in R(D_u\mathcal{F}(0, u_1))$, $\nabla G(0, 0) = 0$ and $\tilde{\eta}_\lambda(0, 0) = v_1$ is the unique solution of

$$\begin{cases} \nabla \cdot \left[e^{\alpha P(x)}\nabla v\right] = 0, & x \in \Omega, \\ \partial_{\vec{n}} v = r(x)b(u_1)u_1, & x \in \partial\Omega, \\ \int_\Omega v(x)\mathrm{d}x = 0. \end{cases} \tag{5.10}$$

Moreover, we can derive that $G_{\lambda\sigma}(0, 0) = G_{\sigma\sigma}(0, 0) = 0$, which implies that $\text{Hess}(G) = 0$. Thus one can not apply Lemma 2.5 of Liu, Shi and Wang [39] to G. Meanwhile, the assumption (**H4**) in Theorem 4.6 does not hold. To overcome this difficulty, we define

$$\Psi(\lambda, \sigma) = \int_{\partial\Omega} e^{\alpha P(x)} r(x) b(u_1 + \sigma + \tilde{\eta}(\lambda, \sigma)) \cdot (u_1 + \sigma + \tilde{\eta}(\lambda, \sigma))\mathrm{d}S. \tag{5.11}$$

Thus, solving (5.2) is transformed to find zero of $\Psi(\lambda, \sigma) = 0$ since we look for nontrivial solution to (5.2) with $\lambda > 0$. Notice that $\Psi \in C^2$ for $(\lambda, \sigma) \in \mathcal{U}$ and $\Psi(0, 0) =$

$u_1 b(u_1) \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S = 0$. Then we have the following results regarding the existence of nontrivial solution of (5.2) bifurcating from Γ_{u_1}.

Lemma 5.2. *Suppose that* **(A1)** *and* **(A2)** *hold, and* $r : \partial\Omega \to \mathbb{R}$ *is sign-changing on* $\partial\Omega$ *and satisfies* (5.4). *The following statements are valid:*

(i) The point $(0, \bar{u}_1)$, *where* $0 < \bar{u}_1 < 1$ *and* $b(\bar{u}_1) + b'(\bar{u}_1)\bar{u}_1 = 0$, *is a bifurcation point of* (5.2) *with respect to* Γ_{u_1}, *and the solutions of* (5.2) *near* $(0, \bar{u}_1)$ *consist exactly of the curve* Γ_{u_1} *and*

$$S_1 = \{(\lambda_1(s), u_1(s)) : s \in (-\delta, \delta) \subset \mathbb{R}\},$$

where $\lambda_1(s)$ *and* $u_1(s) = \bar{u}_1 + \sigma_1(s) + \tilde{\eta}(\lambda_1(s), \sigma_1(s))$ *are* C^1 *functions such that* $\lambda_1(0) = \sigma_1(0) = 0$, $\lambda_1'(0) = 1$ *and* $\sigma_1'(0) = -\dfrac{\bar{u}_1 b(\bar{u}_1) \int_{\partial\Omega} e^{\alpha P(x)} r(x)[v_1]^2 \mathrm{d}S}{2\int_{\partial\Omega} e^{\alpha P(x)} |\nabla v_1|^2 \mathrm{d}S}$, *where* $v_1 \in X_1$ *is the unique solution of* (5.10) *with* $u_1 = \bar{u}_1$.

(ii) If $b(u_1) + b'(u_1)u_1 \neq 0$ *with* $0 < u_1 < 1$, *then* $(0, u_1)$ *is not a bifurcation point of* (5.2) *on* Γ_{u_1}.

(iii) There is no nonconstant solution of (5.2) *near* $(0, 0)$ *and* $(0, 1)$.

Proof. 1. For part (i), we note that $\left[ub(u)\right]'\big|_{u=0} = b(0) > 0$, $\left[ub(u)\right]'\big|_{u=1} = b'(1) < 0$ and $\left[ub(u)\right]'' = 2b'(u) + ub''(u) < 0$ for $u \in (0, 1)$, then there exists a unique $\bar{u}_1 \in (0, 1)$ satisfying $b(\bar{u}_1) + b'(\bar{u}_1)\bar{u}_1 = 0$. From the information of the partial derivatives of $\tilde{\eta}$ at $(0, 0)$, by (5.4), we calculate that

$$\begin{aligned}
&\Psi_\lambda(0,0) = 0, \quad \Psi_\sigma(0,0) = 0,\\
&\Psi_{\lambda\lambda}(0,0) = \left[2b'(u_1) + u_1 b''(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) \left[\tilde{\eta}_\lambda(0,0)\right]^2 \mathrm{d}S\\
&\qquad = \left[2b'(u_1) + u_1 b''(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) \left[v_1\right]^2 \mathrm{d}S,\\
&\Psi_{\lambda\sigma}(0,0) = \left[2b'(u_1) + u_1 b''(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) \left[1 + \tilde{\eta}_\sigma(0,0)\right] \tilde{\eta}_\lambda(0,0) \mathrm{d}S\\
&\qquad = \left[2b'(u_1) + u_1 b''(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) v_1(x) \mathrm{d}S,\\
&\Psi_{\sigma\sigma}(0,0) = \left[2b'(u_1) + u_1 b''(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) \left[1 + \tilde{\eta}_\sigma(0,0)\right]^2 \mathrm{d}S = 0.
\end{aligned}$$

At the point $(0, \bar{u}_1)$, by (5.10), we can infer that

$$\int_{\partial\Omega} e^{\alpha P(x)} r(x) v_1(x) \mathrm{d}S = \frac{1}{\bar{u}_1 b(\bar{u}_1)} \int_{\Omega} e^{\alpha P(x)} |\nabla v_1|^2 \mathrm{d}x > 0,$$

which implies that

$$\Psi_{\lambda\sigma}(0,0) = \left[2b'(u_1) + u_1 b''(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) v_1(x) \mathrm{d}S < 0.$$

Thus we conclude that the Hessian matrix of Ψ at $(0,0)$ and $u_1 = \bar{u}_1$ is indefinite and non-degenerate since

$$\det \mathrm{Hess}(\Psi) = \Psi_{\lambda\lambda}(0,0) \cdot \Psi_{\sigma\sigma}(0,0) - \Psi_{\lambda\sigma}(0,0)^2 = -\Psi_{\lambda\sigma}(0,0)^2 < 0.$$

It follows from Lemma 2.5 of Liu, Shi and Wang [39] that the solution set of $\Psi(\lambda,\sigma) = 0$ near $(\lambda,\sigma) = (0,0)$ is a pair of intersecting curves. Consequently, the solution set of (5.2) near $(\lambda, u) = (0,0)$ is precisely a pair of intersecting curves which solve $\Psi(\lambda,\sigma) = 0$ and has the form

$$\{(\lambda_i(s), u_i(s)) : |s| < \delta\} \ \ (i = 1, 2)$$

for some $\delta > 0$ with $u_i(s) = \sigma_i(s) + \tilde{\eta}_i(\lambda_i(s), u_i(s))$, where $(\lambda_1'(0), \sigma_1'(0))$ and $(\lambda_2'(0), \sigma_2'(0))$ are non-zero linear independent solutions of

$$\Psi_{\lambda\lambda}(0,0)x^2 + 2\Psi_{\lambda\sigma}(0,0)xy = 0.$$

Now we choose $(\lambda_1'(0), \sigma_1'(0)) = (1, -\frac{\Psi_{\lambda\lambda}(0,0)}{2\Psi_{\lambda\sigma}(0,0)})$, $(\lambda_2'(0), \sigma_2'(0)) = (0,1)$ such that the curve $\{(\lambda_2(s), u_2(s))\}$ coincides with the trivial solution set Γ_{u_1}, while the curve $S_1 = \{(\lambda_1(s), u_1(s))\}$ must be distinct from Γ_{u_1}. This proves part (i).

2. For part (ii), we suppose that $u_1 \in (0,1)$ and $b(u_1) + b'(u_1)u_1 \neq 0$. Notice that $\Psi(0,0) = 0$ and

$$\begin{aligned}\Psi_\lambda(0,0) &= \left[b(u_1) + u_1 b'(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) \tilde{\eta}_\lambda(0,0) \mathrm{d}S \\ &= \left[b(u_1) + u_1 b'(u_1)\right] \int_{\partial\Omega} e^{\alpha P(x)} r(x) v_1(x) \mathrm{d}S \\ &= \frac{b(u_1) + u_1 b'(u_1)}{u_1 b(u_1)} \int_{\Omega} e^{\alpha P(x)} |\nabla v_1|^2 \mathrm{d}x \neq 0.\end{aligned}$$

It follows from the implicit function theorem that the unique solution set of $\Psi(\lambda,\sigma) = 0$ in a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of $(0,0)$ consists of the graph of a C^2 function $\lambda = \lambda(\sigma)$ for $(\lambda,\sigma) \in \mathcal{U}_1$, which satisfies $\lambda(0) = 0$. Recalling the equation (5.2), we see that for $\lambda = 0$, $u = u_1 + \sigma + \tilde{\eta}(0,\sigma)$ must be a constant. This combined with $\tilde{\eta} \in X_1$ means that $\tilde{\eta}(0,\sigma) \equiv 0$. Thus, we obtain that

$$\Psi(0,\sigma) = \int_{\partial\Omega} e^{\alpha P(x)} r(x) b(u_1 + \sigma) \cdot (u_1 + \sigma) \mathrm{d}S = b(u_1 + \sigma) \cdot (u_1 + \sigma) \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S = 0.$$

By the uniqueness of solution of $\Psi(\lambda,\sigma)=0$ in $\mathcal{U}_1$, we must have $\lambda(\sigma)=0$ near $\sigma=0$. Therefore, the solution set of (5.2) near $(0,u_1)$ is only the branch Γ_{u_1} and no bifurcation occurs. This proves part (ii).

3. Finally, for part (iii), we let $\hat{u}_1\in\{0,1\}$. In this situation, we still have $\tilde{\eta}(0,\sigma)\equiv 0$, which implies that $\tilde{\eta}_\sigma(0,0)=\tilde{\eta}_{\sigma\sigma}(0,0)=0$. Moreover, since $v_1\in X_1$, from (5.10) with $u_1\in\{0,1\}$, one can derive that $\tilde{\eta}_\lambda(0,0)=v_1\equiv 0$. By (5.8), we calculate that

$$\begin{aligned}0&=(I-P)\big(D_{\lambda u}\mathcal{F}(0,\hat{u}_1)\big[1+\tilde{\eta}_\sigma(0,0)\big]+D_{uu}\mathcal{F}(0,\hat{u}_1)\big[1+\tilde{\eta}_\sigma(0,0),\tilde{\eta}_\lambda(0,0)\big]\\&\quad+D_u\mathcal{F}(0,\hat{u}_1)\big[\tilde{\eta}_{\lambda\sigma}(0,0)\big]\big)\\&=D_{\lambda u}\mathcal{F}(0,\hat{u}_1)\big[1\big]+D_u\mathcal{F}(0,\hat{u}_1)\big[\tilde{\eta}_{\lambda\sigma}(0,0)\big],\end{aligned}$$

and

$$\begin{aligned}0&=(I-P)\big(D_{\lambda u}\mathcal{F}(0,\hat{u}_1)\big[\tilde{\eta}_\lambda(0,0)\big]+D_{uu}\mathcal{F}(0,\hat{u}_1)\big[\tilde{\eta}_\lambda(0,0),\tilde{\eta}_\lambda(0,0)\big]\\&\quad+D_u\mathcal{F}(0,\hat{u}_1)\big[\tilde{\eta}_{\lambda\lambda}(0,0)\big]\big)\\&=D_u\mathcal{F}(0,\hat{u}_1)\big[\tilde{\eta}_{\lambda\lambda}(0,0)\big].\end{aligned}$$

Note that

$$\big\langle l, D_{\lambda u}\mathcal{F}(0,\hat{u}_1)\big[1\big]\big\rangle=[b(\hat{u}_1)+\hat{u}_1b'(\hat{u}_1)]\int_{\partial\Omega}e^{\alpha P(x)}r(x)\mathrm{d}S=0.$$

Then $D_{\lambda u}\mathcal{F}(0,\hat{u}_1)\big[1\big]\in R(D_u\mathcal{F}(0,\hat{u}_1))$. We now can infer that $\tilde{\eta}_{\lambda\lambda}(0,0)=0$ and $\tilde{\eta}_{\lambda\sigma}(0,0)=v_2\in X_1$ is the unique solution of

$$\begin{cases}\nabla\cdot\big[e^{\alpha P(x)}\nabla v\big]=0, & x\in\Omega,\\ \partial_{\vec{n}}v=r(x)[b(\hat{u}_1)+\hat{u}_1b'(\hat{u}_1)], & x\in\partial\Omega.\end{cases}\tag{5.12}$$

Hence, we have that $\Psi_\lambda(0,0)=\Psi_{\lambda\lambda}(0,0)=0$ and

$$\Psi_{\lambda\sigma}(0,0)=[b(\hat{u}_1)+\hat{u}_1b'(\hat{u}_1)]\int_{\partial\Omega}e^{\alpha P(x)}r(x)v_2(x)\mathrm{d}S=\int_\Omega e^{\alpha P(x)}|\nabla v_2|^2\mathrm{d}x\neq 0.$$

This leads to that det Hess$(\Psi)<0$. Likewise, the intersecting solution curves of $\Psi(\lambda,\sigma)=0$ near $(0,0)$ coincide with the trivial solution sets Γ_0,Γ_{u_1} for $\hat{u}_1=0$, and Γ_1,Γ_{u_1} for $\hat{u}_1=1$. This completes the proof. □

Remark 5.3. Lemma 5.2 implies that the point $(0,\bar{u}_1)$, where $0<\bar{u}_1<1$ and $b(\bar{u}_1)+b'(\bar{u}_1)\bar{u}_1=0$, is the unique bifurcation point of (5.2) with respect to Γ_{u_1}.

The following lemma can help us getting better understanding of the bifurcation and stability structures of nonconstant steady states to (5.1).

Lemma 5.4. *Let u_λ be a nontrivial steady state of (5.1) for $\lambda > 0$. Then the operator $D_u\mathcal{F}(\lambda, u_\lambda) : W_l^2(\Omega) \to L^l(\Omega) \times W_l^{1-1/l}(\partial\Omega), l > N$, given by*

$$D_u\mathcal{F}(\lambda, u_\lambda)[v] = \Big(\nabla \cdot \big[e^{\alpha P(x)}\nabla v\big], \partial_{\vec{n}} v - \lambda r(x)[b(u_\lambda) + u_\lambda b'(u_\lambda)]v\Big)$$

for all $v \in W_l^2(\Omega)$, is an injective mapping.

Proof. Suppose to the contrary that $D_u\mathcal{F}(\lambda, u_\lambda)$ is not injective. Then the elliptic problem

$$\begin{cases} \nabla \cdot \big[e^{\alpha P(x)}\nabla \xi\big] = 0, & x \in \Omega, \\ \partial_{\vec{n}}\xi = \lambda r(x)[b(u_\lambda) + u_\lambda b'(u_\lambda)]\xi, & x \in \partial\Omega \end{cases} \tag{5.13}$$

admits a nontrivial solution $\xi \in W_l^2(\Omega)$. By elliptic regularity (refer to [44]) ξ is a classical solution of above equation. By applying the maximum principle and Hopf's Lemma, we obtain that $u_\lambda > 0$ and $1 - u_\lambda > 0$ on $\overline{\Omega}$. Thus $b(u_\lambda) > 0$ on $\overline{\Omega}$.

Define a function $\zeta = \frac{\xi}{u_\lambda b(u_\lambda)}$. By letting $\xi = \zeta u_\lambda b(u_\lambda)$ into (5.13), we have

$$\begin{aligned} \nabla \cdot \big[e^{\alpha P(x)}\nabla \xi\big] &= \nabla \cdot \big[e^{\alpha P(x)}\nabla\big(\zeta u_\lambda b(u_\lambda)\big)\big] \\ &= e^{\alpha P(x)}\Big[\nabla \cdot \nabla\big(\zeta u_\lambda b(u_\lambda)\big) + \nabla P \cdot \nabla\big(\zeta u_\lambda b(u_\lambda)\big)\Big] \\ &= e^{\alpha P(x)}\Big[u_\lambda b(u_\lambda)\Delta\zeta + 2\nabla\zeta \cdot \nabla\big(u_\lambda b(u_\lambda)\big) + \zeta\Delta\big(u_\lambda b(u_\lambda)\big) \\ &\qquad + u_\lambda b(u_\lambda)\nabla P \cdot \nabla\zeta + \zeta\nabla P \cdot \nabla\big(u_\lambda b(u_\lambda)\big)\Big] = 0 \text{ in } \Omega, \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{\partial\big(\zeta u_\lambda b(u_\lambda)\big)}{\partial\vec{n}} - \lambda r(x)[b(u_\lambda) + u_\lambda b'(u_\lambda)]\zeta u_\lambda b(u_\lambda) \\ &= u_\lambda b(u_\lambda)\frac{\partial\zeta}{\partial\vec{n}} + \big[b(u_\lambda) + u_\lambda b'(u_\lambda)\big]\zeta\frac{\partial u_\lambda}{\partial\vec{n}} - \lambda r(x)[b(u_\lambda) + u_\lambda b'(u_\lambda)]\zeta u_\lambda b(u_\lambda) \\ &= u_\lambda b(u_\lambda)\frac{\partial\zeta}{\partial\vec{n}} \text{ in } \partial\Omega. \end{aligned}$$

Note that by **(A1)**, there holds

$$\begin{aligned} & e^{\alpha P(x)}\Big[\Delta\big(u_\lambda b(u_\lambda)\big) + \nabla P \cdot \nabla\big(u_\lambda b(u_\lambda)\big)\Big] \\ &= \nabla \cdot \big[e^{\alpha P(x)}\nabla\big(u_\lambda b(u_\lambda)\big)\big] \\ &= \nabla \cdot \big[\big(b(u_\lambda) + u_\lambda b'(u_\lambda)\big)e^{\alpha P(x)}\nabla u_\lambda\big] \\ &= \big(b(u_\lambda) + u_\lambda b'(u_\lambda)\big)\nabla \cdot \big[e^{\alpha P(x)}\nabla u_\lambda\big] + \big(2b'(u_\lambda) + u_\lambda b''(u_\lambda)\big)e^{\alpha P(x)}|\nabla u_\lambda|^2 \\ &= \big(2b'(u_\lambda) + u_\lambda b''(u_\lambda)\big)e^{\alpha P(x)}|\nabla u_\lambda|^2 \le 0. \end{aligned}$$

Then ζ satisfies an elliptic equation

$$\begin{cases} \Delta\zeta + \sum_{i=1}^{N} b_i(x)\zeta_{x_i} + c(x)\zeta = 0, & x \in \Omega, \\ \partial_{\vec{n}}\zeta = 0, & x \in \partial\Omega, \end{cases}$$

where the coefficients b_i and c are smooth and $c(x) \leq 0$ for $x \in \Omega$. Combining the maximum principle and the Hopf's Lemma, we can derive that $\zeta \equiv 0$, a contradiction. □

We can infer from Lemma 5.4 and the implicit function theorem that there is no secondary bifurcation of steady state solution to (5.1), and have the following result.

Theorem 5.5. *Suppose that* **(A1)** *and* **(A2)** *hold, and $r : \partial\Omega \to \mathbb{R}$ is sign-changing on $\partial\Omega$ and satisfies* (5.4). *Then for each $\lambda > 0$ there is a unique nonconstant solution u_λ to* (5.2)*, which coincides with a global smooth curve bifurcating from Γ_{u_1}. Moreover, the bifurcating solution curve is globally parameterized in λ. That is, the mapping*

$$(0, +\infty) \ni \lambda \longmapsto u_\lambda \in \left[W_l^2(\Omega) \cap \mathfrak{X}\right] \; (l > N),$$

is smooth.

Proof. It follows from Lemma 5.2 that there is a local smooth solution curve S_1 of (5.2) bifurcating from Γ_{u_1} at the point $(0, \bar{u}_1)$ with $0 < \bar{u}_1 < 1$ and $b(\bar{u}_1) + b'(\bar{u}_1)\bar{u}_1 = 0$. Clearly, this bifurcating solution is nonconstant. By applying the implicit function theorem, Lemma 5.2 and the continuation of S_1 in λ, we can extend such local curve to a global smooth solution curve, denoted by $\mathcal{S}$, parameterized by λ, where S_1 coincides with $\mathcal{S}$ for λ near 0. In view of Lemma 5.2 (iii), since there is no secondary bifurcation of steady state solution to (5.1), the obtained global solution curve is defined for all $\lambda > 0$.

For the uniqueness, it is true for λ near 0. Suppose that there exists $\hat{\lambda} \gg 0$ such that $u_{\hat{\lambda}} \in \mathfrak{X} \setminus \mathcal{S}$ is a nonconstant solution to (5.2). By the argument as above, since no secondary bifurcation occurs, one can get a global smooth solution curve $\hat{\mathcal{S}}$ to (5.2), in which $(\hat{\lambda}, u_{\hat{\lambda}}) \in \hat{\mathcal{S}}$ and $\hat{\mathcal{S}}$ does not coincide with S_1 for λ near 0. This is impossible by the uniqueness of S_1 near $(0, \bar{u}_1)$. Thus the existence of a unique nonconstant steady state solution u_λ of (5.2) for $\lambda > 0$ is obtained. Meanwhile, we see that the mapping $(0, +\infty) \ni \lambda \longmapsto u_\lambda \in \left[W_l^2(\Omega) \cap \mathfrak{X}\right](l > N)$, is smooth. The proof is completed. □

In the following, we give a stability analysis for the steady state solutions of (5.1). From the previous argument, we know that (5.1) admits the constant steady states $u = 0$ and $u = 1$, and a unique nonconstant steady state u_λ define for all $\lambda > 0$. Their stability can be classified as follows.

Theorem 5.6. *Suppose that* **(A1)** *and* **(A2)** *hold, and $r : \partial\Omega \to \mathbb{R}$ is sign-changing on $\partial\Omega$ and satisfies* (5.4). *Let u_λ be the unique nonconstant solution to* (5.2).

(i) The constant steady states $u = 0$ and $u = 1$ of (5.1) *are both unstable for all $\lambda > 0$.*
(ii) For any $\lambda > 0$, the steady state u_λ of (5.1) *is globally asymptotically stable, i.e., for any initial value $u_0 \in \mathfrak{X}$, satisfying $u_0 \not\equiv 0, 1$, the solution $u(\cdot, t; u_0)$ of* (5.1) *converges to u_λ in the W_l^1-norm, $l > N$.*

This theorem will be proved by several steps, and the proof for part (i) is as follows.

Proof of Theorem 5.6 (i). Linearizing the equation (5.1) at $\bar{u}$ ($\bar{u} = 0$ or $\bar{u} = 1$), we obtain the corresponding eigenvalue problem

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi \right] = \mu \psi, & x \in \Omega, \\ \partial_{\vec{n}} \psi - \lambda r(x) \left[b(\bar{u}) + \bar{u} b'(\bar{u}) \right] \psi = \mu \psi, & x \in \partial\Omega. \end{cases} \tag{5.14}$$

Since $\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S = 0$, following the proof of Theorem 3.4 (ii), we have that no matter for $\bar{u} = 0$ or for $\bar{u} = 1$, the principal eigenvalue μ_1 of (5.14) must be negative. Then by Proposition 3.2, the constant steady states $u = 0$ and $u = 1$ of (5.1) are both unstable. □

We now provide a stronger conclusion regarding the instability property of the constant steady states $u = 0$ and $u = 1$ of (5.1), which shows that any nontrivial semi-orbit of the dynamical system generated by (5.1) in $\mathfrak{X}$ would not converge to $u = 0$ or $u = 1$.

Lemma 5.7. *Suppose that* **(A1)** *and* **(A2)** *hold, and* $r : \partial\Omega \to \mathbb{R}$ *is sign-changing on* $\partial\Omega$ *and satisfies* (5.4). *Then for any initial value* $u_0 \in \mathfrak{X}$, *satisfying* $u_0 \not\equiv 0, 1$, *the solution* $u(\cdot, t; u_0)$ *of* (5.1) *has the properties that*

$$\|u(\cdot, t; u_0)\|_{W_l^1(\Omega)} \nrightarrow 0 \quad \textit{and} \quad \|u(\cdot, t; u_0) - 1\|_{W_l^1(\Omega)} \nrightarrow 0$$

as $t \to +\infty$, *where* $l > N$.

Proof. We only prove the nonexistence of nontrivial positive semi-orbit converging to $u = 0$, and that for $u = 1$ can be proved similarly.

By way of contradiction, we suppose that there exists $u_0 \in \mathfrak{X}$ with $u_0 \not\equiv 0, 1$, such that

$$\|u(\cdot, t; u_0)\|_{W_l^1(\Omega)} \longrightarrow 0, \text{ as } t \to +\infty,$$

for some $l > N$. Let $\mu_1 = \mu_1(\lambda)$ be the principal eigenvalue of the problem

$$\begin{cases} -e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla \psi \right] = \mu \psi, & x \in \Omega, \\ \partial_{\vec{n}} \psi - \lambda r(x) b(0) \psi = \mu \psi, & x \in \partial\Omega, \end{cases}$$

and ψ_1 be the eigenfunction corresponding to μ_1 normalized by $\|\psi_1\|_{L^2(\Omega)} = 1$. By the proof of Theorem 5.6 (i), it holds that $\mu_1 < 0$. Define $\Phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\Phi(t) = \int_\Omega e^{\alpha P(x)} u(\cdot, t; u_0) \psi_1 \mathrm{d}x.$$

We claim that Φ *is strictly increasing for sufficiently large* t.

For the proof of above claim, we write $u(t) := u(\cdot, t; u_0)$ and have that

$$\begin{aligned}\frac{d\Phi(t)}{dt} &= \int_\Omega e^{\alpha P(x)} u_t \psi_1 \mathrm{d}x = \int_\Omega \psi_1 \nabla\cdot\left[e^{\alpha P(x)}\nabla u\right]\mathrm{d}x \\ &= \int_\Omega u(t)\nabla\cdot\left[e^{\alpha P(x)}\nabla\psi_1\right]\mathrm{d}x - \int_{\partial\Omega} u(t)e^{\alpha P(x)}\partial_{\vec{n}}\psi_1 \mathrm{d}S + \int_{\partial\Omega}\psi_1 e^{\alpha P(x)}\partial_{\vec{n}} u \mathrm{d}S \\ &= -\mu_1\left(\int_\Omega e^{\alpha P(x)}\psi_1 u(t)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\psi_1 u(t)\mathrm{d}S\right) \\ &\quad + \lambda\int_{\partial\Omega} e^{\alpha P(x)}\psi_1[b(u(t)) - b(0)]u(t)\mathrm{d}S.\end{aligned}$$

Notice that by the maximum principle and Hopf's Lemma, $0 < u(t) < 1$ on $\overline{\Omega}$ for all $t > 0$. Moreover, we see that

$$\|u(t)\|_{C(\overline{\Omega})} \longrightarrow 0, \text{ as } t \to +\infty, \tag{5.15}$$

as the embedding $W_l^1(\Omega) \hookrightarrow C(\overline{\Omega})$ holds for $l > N$. Combining (5.15) and the continuation of $b(u)$ at $u = 0$, and using $\mu_1 < 0$, one infer that there is $t_0 > 0$ large enough such that

$$\begin{aligned}\frac{d\Phi(t)}{dt} &= -\mu_1\left(\int_\Omega e^{\alpha P(x)}\psi_1 u(t)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\psi_1 u(t)\mathrm{d}S\right) \\ &\quad + \lambda\int_{\partial\Omega} e^{\alpha P(x)}\psi_1[b(u(t)) - b(0)]u(t)\mathrm{d}S > 0\end{aligned}$$

for $t \geq t_0$. This proves the claim.

Next we choose $t_* > t_0$ such that $\|u(t_*)\|_{C(\overline{\Omega})} < \inf_{\overline{\Omega}} u(t_0)$. It follows from above claim that $\Phi(t_0) < \Phi(t_*)$, consequently, $\int_\Omega e^{\alpha P(x)}\left[u(t_0) - \inf_{\overline{\Omega}} u(t_0)\right]\psi_1 \mathrm{d}x < 0$, which is a contradiction. The proof is finished. □

Define a function $\mathcal{J}_\lambda : \mathfrak{X} \to \mathbb{R}$ as

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_\Omega e^{\alpha P(x)}|\nabla u|^2 \mathrm{d}x - \lambda\int_{\partial\Omega} e^{\alpha P(x)} r(x)\bar{B}(u)\mathrm{d}S, \tag{5.16}$$

where $\bar{B} = \int_0^u \xi b(\xi)\mathrm{d}\xi$. Then $\mathcal{J}_\lambda$ is a Lyapunov function for the dynamical system generated by (5.1) for all $\lambda > 0$, which is a gradient system (see [2]). Further, $\mathcal{J}_\lambda$ decreases along the semi-orbits except at steady states of (5.1). Meanwhile, the corresponding semi-orbit in gradient system would converge to the set $\mathscr{C}$ of steady state solutions of (5.1) as time goes to infinite (see [27]). For more details, one can see from [27, Theorems 4.3.3 and 4.3.4] that, for any initial value $u_0 \in \mathfrak{X}$, the omega limit set $\omega(u_0)$ associated with the dynamical system generated by (5.1), which is nonempty, compact, connected and invariant, is contained in $\mathscr{C}$. Then we have the following result.

Lemma 5.8. *Suppose that (A1) and* **(A2)** *hold, and $r : \partial\Omega \to \mathbb{R}$ is sign-changing on $\partial\Omega$ and satisfies* (5.4). *Then the nonconstant steady state u_λ of* (5.1) *is a global minimizer of the energy functional J_λ in $\mathfrak{X}$ for all $\lambda > 0$.*

Proof. It is known that critical points of $\mathcal{J}_\lambda$ defined by (5.16) are weak solutions of (5.2) for $\lambda > 0$. Firstly, we show that $\mathcal{J}_\lambda|_{\mathfrak{X}}$ has a global minimizer in $\mathfrak{X}$. Define

$$\{r \geq 0\} := \{x \in \partial\Omega : r(x) \geq 0\} \text{ and } \{r \leq 0\} := \{x \in \partial\Omega : r(x) \leq 0\}.$$

Then for any $u \in \mathfrak{X}$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\int_\Omega e^{\alpha P(x)}|\nabla u|^2 \mathrm{d}x - \lambda \int_{\{r\geq 0\}} e^{\alpha P(x)} r(x)\bar{B}(u)\mathrm{d}S - \lambda \int_{\{r\leq 0\}} e^{\alpha P(x)} r(x)\bar{B}(u)\mathrm{d}S \\ &\geq \frac{1}{2}e^{\alpha \min_{\overline{\Omega}} P(x)}\|u\|^2_{H^1(\Omega)} - \frac{1}{2}e^{\alpha \min_{\overline{\Omega}} P(x)}\int_\Omega u^2 \mathrm{d}x - \lambda \int_{\{r\geq 0\}} e^{\alpha P(x)} r(x)\int_0^u \xi b(\xi)\mathrm{d}\xi \mathrm{d}S \\ &\geq \frac{1}{2}e^{\alpha \min_{\overline{\Omega}} P(x)}\|u\|^2_{H^1(\Omega)} - K, \end{aligned}$$

where $K = K(\lambda, \Omega, r, b) > 0$, which means that $\mathcal{J}_\lambda$ is bounded from below in $\mathfrak{X}$. Thus one can define the infimum $\chi := \inf_{u\in\mathfrak{X}} \mathcal{J}_\lambda$ and take a minimizing sequence $\{u_m\} \subset \mathfrak{X}$ such that $\mathcal{J}_\lambda(u_m) \to \chi$ as $m \to \infty$.

In view of the estimate from below for $\mathcal{J}_\lambda$, we see that $\{u_m\}$ is bounded in $H^1(\Omega)$. By passing to a subsequence of $\{u_m\}$ if necessary, there exists $\hat{u}_\lambda \in H^1(\Omega)$ satisfying that, as $m \to \infty$,

- $u_m \rightharpoonup \hat{u}_\lambda$ in $H^1(\Omega)$,
- $u_m \to \hat{u}_\lambda$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$,
- $u_m \to \hat{u}_\lambda$ a.e. in $\partial\Omega$,

which also implies that $\hat{u}_\lambda \in \mathfrak{X}$. Noticing that $\|\hat{u}_\lambda\|^2_{H^1(\Omega)} \leq \liminf_{m\to\infty} \|u_m\|^2_{H^1(\Omega)}$, we obtain

$$\chi \leq \mathcal{J}_\lambda(\hat{u}_\lambda) \leq \liminf_{m\to\infty} \mathcal{J}_\lambda(u_m) = \chi,$$

thus, $\mathcal{J}_\lambda(\hat{u}_\lambda) = \chi$, and then $\hat{u}_\lambda$ is a global minimizer of $\mathcal{J}_\lambda$ in $\mathfrak{X}$ for all $\lambda > 0$. On the other hand, by Theorem 5.5, we obtain $\mathscr{C} = \{0, u_\lambda, 1\}$ for all λ. So $\hat{u}_\lambda$ must be one of $0, u_\lambda$ and 1.

In the following, we exclude the possibility for $\hat{u}_\lambda = 0$ or $\hat{u}_\lambda = 1$. In fact, let μ_1^0 be the principal eigenvalue of (5.14) with $\bar{u} = 0$, and ψ_1^0 be the corresponding positive eigenfunction satisfying $\|\psi_1^0\|_{L^2(\Omega)} = 1$. Since (5.4) holds, then $\mu_1^0 < 0$. For a fixed $\delta > 0$, there holds that

$$
\begin{aligned}
\mathcal{J}_\lambda(\delta\psi_1^0) &= \frac{1}{2}\int_\Omega e^{\alpha P(x)}|\nabla(\delta\psi_1^0)|^2 \mathrm{d}x - \lambda\int_{\partial\Omega} e^{\alpha P(x)} r(x)\bar{B}(\delta\psi_1^0)\mathrm{d}S \\
&= \frac{\delta^2}{2}\left[\int_\Omega e^{\alpha P(x)}|\nabla\psi_1^0|^2\mathrm{d}x - \lambda\int_{\partial\Omega} e^{\alpha P(x)} r(x)b(0)\left(\psi_1^0\right)^2\mathrm{d}S\right] \\
&\quad - \frac{\lambda\delta^3}{3}\int_{\partial\Omega} e^{\alpha P(x)} r(x)b(\theta)\left(\psi_1^0\right)^3\mathrm{d}S \\
&= \frac{\delta^2}{2}\mu_1^0\left[\int_\Omega e^{\alpha P(x)}\left(\psi_1^0\right)^2\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\left(\psi_1^0\right)^2\mathrm{d}S\right] - \frac{\lambda\delta^3}{3}\int_{\partial\Omega} e^{\alpha P(x)} r(x)b(\theta)\left(\psi_1^0\right)^3\mathrm{d}S,
\end{aligned}
$$

where $\theta(x)$ is a function between 0 and $\delta\psi_1^0(x)$ for $x \in \partial\Omega$. Then it can be inferred that $\mathcal{J}_\lambda(\delta\psi_1^0) < 0$ for $\delta > 0$ small enough. This is to say $J_\lambda(\hat{u}_\lambda) < 0$. Since $\mathcal{J}_\lambda(0) = \mathcal{J}_\lambda(1) = 0$ for all $\lambda > 0$, and $\delta\psi_1^0 \in \mathfrak{X}$ for $\delta > 0$ small, it must be $\hat{u}_\lambda = u_\lambda$, consequently, u_λ is a global minimizer of the energy functional J_λ in $\mathfrak{X}$ for all $\lambda > 0$. The proof is completed. □

In this position, we can complete the proof of Theorem 5.6 (ii).

Proof of Theorem 5.6 (ii). Linearizing equation (5.1) at the steady state u_λ, we obtain the eigenvalue problem

$$
\begin{cases}
-e^{-\alpha P(x)}\nabla\cdot\left[e^{\alpha P(x)}\nabla\psi\right] = \mu\psi, & x \in \Omega, \\
\partial_{\vec{n}}\psi - \lambda r(x)\left[b(u_\lambda) + u_\lambda b'(u_\lambda)\right]\psi = \mu\psi, & x \in \partial\Omega.
\end{cases} \tag{5.17}
$$

The principal eigenvalue μ_1 of (5.17) has the variational characterization

$$
\mu_1 = \inf_{\psi\in H^1(\Omega)} \frac{\int_\Omega e^{\alpha P(x)}|\nabla\psi|^2\mathrm{d}x - \lambda\int_{\partial\Omega} e^{\alpha P(x)} r(x)\left[b(u_\lambda) + u_\lambda b'(u_\lambda)\right]\psi^2\mathrm{d}S}{\int_\Omega e^{\alpha P(x)}\psi^2\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\psi^2\mathrm{d}S}.
$$

Notice that $0 < u_\lambda < 1$ on $\overline{\Omega}$ by the maximum principle, and $W_l^1(\Omega) \hookrightarrow C(\overline{\Omega})$ for $l > N$. Then there is a small ball in the W_l^1-topology, with $l > N$, which is centered at u_λ and contained in $\mathfrak{X}$. Thus there holds that

$$
-\mathcal{J}_\lambda''(u_\lambda)[\psi]^2 \leq 0, \quad \forall\psi \in W_l^1(\Omega), l > N,
$$

since u_λ is local minimizer of the functional $\mathcal{J}_\lambda$ in $W_l^1(\Omega)$, $l > N$, by Lemma 5.8. Hence, we have $\mu_1 \geq 0$. However, it follows from Lemma 5.4 that $\mu_1 = 0$ is not an eigenvalue of (5.17) and thus $\mu_1 > 0$. Now, by Proposition 3.2, u_λ is locally stable. This together with Lemma 5.7 proves Theorem 5.6 (ii). □

5.2. Case (5.5)

In this subsection, we consider the dynamics of (5.1) when (5.5) holds. Note that the trivial solution sets Γ_0 and Γ_{u_1} of (5.2) intersect at the point $(0,0)$, while Γ_1 and Γ_{u_1} intersect at the point $(0,1)$. We first have the following conclusion.

Proposition 5.9. *Suppose that* **(A1)** *and* **(A2)** *hold, and $r : \partial\Omega \to \mathbb{R}$ is sign-changing on $\partial\Omega$ and satisfies* (5.5). *Then there is no nonconstant solution of* (5.2) *near* $(0,0)$ *and* $(0,1)$.

Proof. Since (5.5) holds true, by Theorem 4.7, the solution set of (5.2) near $(0,0)$ is a union of Γ_0 and Γ_{u_1}, which implies the nonexistence of nonconstant solution of (5.2) near $(0,0)$. Secondly, if u_1 is replaced by 1, then Theorem 4.6 can also be applied to the point $(0,1)$ since $f(x,u) \equiv 0$ and

$$\int_{\partial\Omega} e^{\alpha P(x)}[\beta(x,1)+\beta_u(x,1)]\mathrm{d}S = b'(1)\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S \neq 0.$$

Thus we can obtain that the solution set of (5.2) near $(0,1)$ is a union of Γ_1 and Γ_{u_1}, and then there is no nonconstant solution of (5.2) near $(0,1)$. □

It can be seen from Theorem 2.2 that problem (5.6) admits a unique positive principal eigenvalue λ_1 when (5.5) is satisfied, which shows that λ_1 is the unique bifurcation point with respect to Γ_0. The next result concerning the solution set of (5.2) in the range $\lambda \in (0, \lambda_1]$.

Lemma 5.10. *Suppose that $r : \partial\Omega \to \mathbb{R}$ is sign-changing on $\partial\Omega$ and $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S \neq 0$. Then for all $0 < \lambda \le \lambda_1$, the only steady state solutions to* (5.2) *are constant ones.*

Proof. Similar to [46, Theorem 3.1], we can infer that for all λ sufficiently small, (5.1) only has the constant steady states. To prove our aim, we suppose to the contrary that there exists $0 < \hat{\lambda} < \lambda_1$ such that (5.2) has a nonconstant steady state $u_{\hat{\lambda}}$ for $\lambda = \hat{\lambda}$. Since $D_u\mathcal{F}(\hat{\lambda}, u_{\hat{\lambda}})$ is a Fredholm operator with zero index, by Lemma 5.4, we know that $D_u\mathcal{F}(\hat{\lambda}, u_{\hat{\lambda}})$ is a bijection. It then follows from the implicit function theorem that there is an interval $\hat{I} := (\hat{\lambda}-\delta, \hat{\lambda}+\delta)$, $\delta > 0$, and solutions $u_{\hat{\lambda}}(\lambda) \in W_l^2(\Omega)$, $l > N$, of (5.2) for all $\lambda \in \hat{I}$ such that $u_{\hat{\lambda}}(\hat{\lambda}) = u_{\hat{\lambda}}$. One can easily have that, reducing $\delta > 0$ if necessary, $u_{\hat{\lambda}}(\lambda) \in \mathfrak{X} \setminus \{0, 1\}$ are nonconstant solutions of (5.2) for all $\lambda \in \hat{I}$.

Choose a sequence $\{\lambda_{(k)}\} \subset \hat{I}$ satisfying $\lambda_{(k)} \to \hat{\lambda} - \delta$ as $k \to \infty$, and denote $u_{\hat{\lambda}}(\lambda_{(k)})$ by $u_{(k)}$, then

$$\int_{\Omega} e^{\alpha P(x)} |\nabla u_{(k)}|^2 \mathrm{d}x = \lambda_{(k)} \int_{\partial\Omega} e^{\alpha P(x)} r(x) b(u_{(k)}) u_{(k)}^2 \mathrm{d}S = O(\lambda_{(k)}).$$

This equality implies that the sequence $\{u_{(k)}\}$ is bounded in $H^1(\Omega)$ and, passing to a subsequence of $\{u_{(k)}\}$ if necessary, there exists $\hat{u} \in H^1(\Omega)$ satisfying that, as $k \to \infty$,

- $u_{(k)} \rightharpoonup \hat{u}$ in $H^1(\Omega)$,
- $u_{(k)} \to \hat{u}$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$,

$$\bullet\ u_{(k)} \to \hat{u} \text{ a.e. in } \partial\Omega.$$

Thus, $\hat{u} \in \mathfrak{X}$ is a weak solution of (5.2) with $\lambda = \hat{\lambda} - \delta$, which is also nonconstant. Now, by proceeding all precious argument for $\hat{u}$, a induction can be applied to obtain a sequence $\{u_{\lambda_{(j)}}\}$ consisting of nonconstant steady states of (5.1), in which $\lambda_{(1)} = \hat{\lambda}$ and $\lambda_{(j)} \to 0$ as $j \to \infty$. This is a contradiction with the fact that (5.1) has no nonconstant steady state for $\lambda > 0$ sufficiently small. □

In what follows, we consider the local bifurcation from Γ_0 at the point $(\lambda_1, 0)$. From the argument in Section 4.1, one has

Theorem 5.11. *Suppose that* **(A1)** *and* **(A2)** *hold, and* $r : \partial\Omega \to \mathbb{R}$ *is sign-changing on* $\partial\Omega$ *and satisfies* (5.5). *Then* (5.6) *has a unique positive principal eigenvalue* λ_1*, which is a bifurcation point with respect to* Γ_0*, and the nonnegative solution set of* (5.2) *near* $(\lambda_1, 0)$ *consists exactly of the curves* Γ_0 *and*

$$S_0 = \{(\lambda_0(s), u_0(s)) : s \in I = (0, \varepsilon) \subset \mathbb{R}^+\},$$

where $\lambda_0(s) = \lambda_1 + z_2(s)$*,* $u_0(s) = s\phi_0 + sz_1(s)$ *are* C^1 *function so that* $z_i(0) = 0, i = 1, 2$*,* ϕ_0 *is the positive eigenfunction associated with* λ_1*. Moreover, the bifurcation occurring at* $(\lambda_1, 0)$ *is transcritical and the bifurcating positive solution of* (5.2) *from* Γ_0 *is locally asymptotically stable.*

Proof. By Proposition 5.1, λ_1 is a bifurcation point of (5.2) with respect to Γ_0. Then it follows from Theorem 4.3 that a positive solution of (5.2) bifurcates from Γ_0 near $(\lambda_1, 0)$. Moreover, by applying (4.12) to S_0, one can derive that

$$\lambda_0'(0) = -2\lambda_1 \cdot \frac{b'(0)}{b(0)} \cdot \frac{\int_\Omega e^{\alpha P(x)} |\nabla\phi_0|^2 \phi_0 \mathrm{d}x}{\int_\Omega e^{\alpha P(x)} |\nabla\phi_0|^2 \mathrm{d}x} > 0. \tag{5.18}$$

Then a transcritical bifurcation occurs at $(\lambda_1, 0)$. This combined with (4.21) shows that the bifurcating positive solution of (5.2) from Γ_0 is stable. □

The above theorem gives a local branch of nonconstant steady state of (5.2) for λ near λ_1. Lemma 5.10 presents the uniqueness result for constant steady states to (5.1) in the range $(0, \lambda_1]$. Then similar to Theorem 5.5, from Lemma 5.4, Proposition 5.9, Lemma 5.10 and the implicit function theorem, we can obtain the following result concerning on a uniqueness result for non-constant steady state, and we omit its proof here.

Theorem 5.12. *Suppose that* **(A1)** *and* **(A2)** *hold, and* $r : \partial\Omega \to \mathbb{R}$ *is sign-changing on* $\partial\Omega$ *and satisfies* (5.5). *Then for each* $\lambda > \lambda_1$ *there is a unique nonconstant solution* u_λ *to* (5.2)*, which coincides with a global smooth curve bifurcating from* Γ_0*. Moreover, the bifurcating solution curve is globally parameterized in* λ*. That is, the mapping*

$$(\lambda_1, +\infty) \ni \lambda \longmapsto u_\lambda \in \left[W_l^2(\Omega) \cap \mathfrak{X}\right] \ (l > N),$$

is smooth.

To end this subsection, we analyze the stability of nonnegative steady states of (5.1). Their stability can be classified as follows.

Theorem 5.13. *Suppose that* **(A1)** *and* **(A2)** *hold, and* $r : \partial\Omega \to \mathbb{R}$ *is sign-changing on* $\partial\Omega$ *and satisfies* (5.5)*. Let* u_λ *be the unique nonconstant solution to* (5.2)*.*

(i) The constant steady state $u = 1$ *of* (5.1) *is unstable for all* $\lambda > 0$*.*
(ii) The constant steady state $u = 0$ *of* (5.1) *is globally asymptotically stable for* $0 < \lambda \le \lambda_1$*, and unstable for* $\lambda > \lambda_1$*.*
(iii) For each $\lambda > \lambda_1$*, the nonconstant steady state* u_λ *of* (5.1) *is globally asymptotically stable.*

Proof. 1. For part (i), since $b'(1) < 0$, one can infer from Theorem 2.2 that if (5.5) is satisfied, then the nonnegative principal eigenvalue of the weighted eigenvalue problem

$$\begin{cases} -e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\phi\right] = 0, & x \in \Omega, \\ \partial_{\vec{n}}\phi = \lambda r(x)b'(1)\phi, & x \in \partial\Omega, \end{cases}$$

is $\tilde{\lambda}_1 = 0$. Linearizing (5.1) at $u = 1$, we obtain the eigenvalue problem

$$\begin{cases} -e^{-\alpha P(x)}\nabla \cdot \left[e^{\alpha P(x)}\nabla\psi\right] = \mu\psi, & x \in \Omega, \\ \partial_{\vec{n}}\psi - \lambda r(x)b'(1)\psi = \mu\psi, & x \in \partial\Omega, \end{cases}$$

which admits a unique principal eigenvalue $\mu_1 = \mu_1(\lambda)$. Since $b'(1) < 0$ and (5.5) holds, following the proof of Theorem 3.4 (ii), we can derive that $\mu_1 < 0$, which together with Proposition 3.2 (ii) means that the constant steady state $u = 1$ of (5.1) is unstable for all $\lambda > 0$. Part (i) is proved.

2. Note that (5.1) is a special case of (1.5) with $f(x,u) \equiv 0$ and $\beta(x,u) = r(x)b(u)$. For part (ii), since (5.5) holds, we have

$$\int_\Omega e^{\alpha P(x)} f(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\alpha P(x)}\beta(x,0)\mathrm{d}S = b(0)\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0.$$

It then follows from Theorem 3.4 that the constant steady state $u = 0$ of (5.1) is locally asymptotically stable for $0 < \lambda < \lambda_1$, and unstable for $\lambda > \lambda_1$. Moreover, when $\lambda \in (0, \lambda_1]$, we see from Lemma 5.10 that the solution set of (5.2) is $\{0, 1\}$. Recall the Lyapunov function $\mathcal{J}_\lambda : \mathfrak{X} \to \mathbb{R}$ for the dynamical system generated by (5.1), defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_\Omega e^{\alpha P(x)}|\nabla u|^2\mathrm{d}x - \lambda\int_{\partial\Omega} e^{\alpha P(x)} r(x)\bar{B}(u)\mathrm{d}S, \quad u \in \mathfrak{X}.$$

Similar to the proof of Lemma 5.8, one obtain that $\mathcal{J}_\lambda$ with $0 < \lambda \le \lambda_1$ has a global minimizer in $\mathfrak{X}$. Then $u \equiv 0$ is the global minimizer of the energy functional $\mathcal{J}_\lambda|_{\mathfrak{X}}$ since $\mathcal{J}_\lambda(0) < \mathcal{J}_\lambda(1)$ and the critical point of $\mathcal{J}_\lambda$ is the weak solution of (5.2). Therefore, from [27], the constant steady state $u = 0$ attracts all orbits dissipating energy, and this completes the proof of part (ii).

3. The proof of part (iii) is similar to that of Theorem 5.6 (ii). We omit it here. □

For the case (5.7), we can similarly obtain that the principal eigenvalue $\tilde{\lambda}_1$ is a bifurcation point with respect to Γ_1, and a local bifurcation occurs at $(\tilde{\lambda}_1, 1)$, which ia transcritical. Moreover, the local bifurcating positive solution can be extended to a global one parameterized by $\lambda \in (\tilde{\lambda}_1, +\infty)$. Likewise, we can give a detailed discussion for the stability of the steady state solutions to (5.1).

6. Application to parabolic equation with sublinear growth and superlinear boundary condition

In this section, we study the dynamics of the following semilinear parabolic equation with a nonlinear boundary condition

$$\begin{cases} u_t = e^{-\alpha P(x)} \nabla \cdot \left[e^{\alpha P(x)} \nabla u \right] + \lambda k(x)(1 - u^{p-1})u, & x \in \Omega, t > 0, \\ \partial_{\vec{n}} u = \lambda r(x) u^q, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) > 0, & x \in \Omega, \end{cases} \tag{6.1}$$

where $p > 1, q > 1$, $\Omega \in \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive parameter, $k(x) \in C^\theta(\overline{\Omega})$ and $r(x) \in C^{1+\theta}(\partial\Omega)$ for some $\theta \in (0, 1)$. The nonnegative steady state solutions of (6.1) satisfy the elliptic equation

$$\begin{cases} \nabla \cdot \left[e^{\alpha P(x)} \nabla u \right] + \lambda e^{\alpha P(x)} k(x)(1 - u^{p-1})u = 0, & x \in \Omega, \\ \partial_{\vec{n}} u = \lambda r(x) u^q, & x \in \partial\Omega. \end{cases} \tag{6.2}$$

Obviously, (6.2) has the trivial solution curves

$$\Gamma_0 := \{(\lambda, 0) : \lambda > 0\}, \text{ and } \Gamma_{u_1} := \{(0, u_1) : u_1 > 0\}.$$

Firstly, from Theorem 3.4, the stability of constant steady state $u = 0$ of (6.1) can be described as follows.

Proposition 6.1. *Suppose that* **(H1)** *holds and $k : \Omega \to \mathbb{R}$ is sign-changing in Ω. Then the following statements hold true:*

(i) If $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x < 0$, then the problem

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla v \right] = \lambda e^{\alpha P(x)} k(x) v, & x \in \Omega, \\ \partial_{\vec{n}} v = 0, & x \in \partial\Omega \end{cases} \tag{6.3}$$

has a unique positive principal eigenvalue λ_1. Moreover, the trivial steady state $u = 0$ of (6.1) is locally asymptotically stable for $0 < \lambda < \lambda_1$, while unstable for $\lambda > \lambda_1$.

(ii) If $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x \geq 0$, then the trivial steady state $u = 0$ of (6.1) is unstable for all $\lambda > 0$.

Proof. Clearly, when $f(x, u) = k(x)(1 - u^{p-1})$ and $\beta(x, u) = r(x) u^{q-1}$, (1.5) is reduced to (6.1). The existence of positive principal eigenvalue λ_1 of (6.3) can be obtained from Theorem 2.2. By checking the condition in Theorem 3.4, one can obtain the conclusions of this proposition directly. □

In the following, we will apply the bifurcation results established in Section 4 to obtain the existence for the nontrivial steady states to (6.1). Notice that the sign of $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x$ determines the stability of the trivial steady state to (6.1). Our argument will be divided into three cases: (1) $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x<0$, (2) $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x=0$ and (3) $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x>0$.

6.1. *Case (1): $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x<0$*

First, we discuss local and global bifurcation from the line of trivial solution curve Γ_0.

Proposition 6.2. *Suppose that* **(H1)** *holds, $k:\Omega\to\mathbb{R}$ is sign-changing in Ω, and $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x<0$. Then*

(i) the positive principal eigenvalue λ_1 of (6.3) is a bifurcation point of (6.2) with respect to Γ_0. Precisely, in a neighborhood of $(\lambda_1,0)$ in $\mathbb{R}^+\times X$, the only positive solution to (1.6) lies in the curve

$$\Sigma_0=\{(\lambda_0(s),u_0(s)):s\in I=(0,\varepsilon)\subset\mathbb{R}^+\},$$

where $\lambda_0(s)=\lambda_1+z_2(s)$, $u_0(s)=s\phi_1+sz_1(s)$ are C^1 function so that $z_i(0)=0, i=1,2$, ϕ_1 is the positive eigenfunction associated with λ_1.

(ii) Denote by $\mathscr{C}$ the connected component of positive solutions of (6.2) which contains the bifurcation curve Σ_0 obtained in part (i). Then the following conclusions are valid:
(ii.a) $\mathscr{C}$ is unbounded in $\mathbb{R}^+\times C(\overline{\Omega})$.
(ii.b) $\overline{\mathscr{C}}\cap\left\{(\lambda,0)|\lambda \text{ is not an eigenvalue of } (6.3)\right\}=\varnothing$.

Proof. When $f(x,u)=k(x)(1-u^{p-1})$ and $\beta(x,u)=r(x)u^{q-1}$, (1.6) is reduced to (6.2). By checking the condition **(H3)** in Theorem 4.3, one can obtain the existence of nontrivial steady state bifurcation directly.

Denote $\mathbb{X}=\{\varphi\in C(\overline{\Omega})|\varphi\geq 0 \text{ on } \overline{\Omega}\}$. Clearly, the interior of $\mathbb{X}$ is given by $\mathring{\mathbb{X}}=\{\varphi\in C(\overline{\Omega})|\varphi>0 \text{ on } \overline{\Omega}\}$. For two positive constants M_1,M_2 and two functions $\varphi_1\in C^\theta(\overline{\Omega})$ and $\varphi_2\in C^{1+\theta}(\partial\Omega)$ with some $\theta\in(0,1)$, consider the following two boundary value problems:

$$\begin{cases}-\nabla\cdot\left[e^{\alpha P(x)}\nabla v\right]+M_1v=\varphi_1, & x\in\Omega,\\ \partial_{\vec{n}}v+M_2v=0, & x\in\partial\Omega,\end{cases}\tag{6.4}$$

and

$$\begin{cases}-\nabla\cdot\left[e^{\alpha P(x)}\nabla w\right]+M_1w=0, & x\in\Omega,\\ \partial_{\vec{n}}w+M_2w=\varphi_2, & x\in\partial\Omega.\end{cases}\tag{6.5}$$

By the theory of linear elliptic equation, we define the solution operators $\mathcal{K}_\Omega:C^\theta(\overline{\Omega})\to C^{2+\theta}(\overline{\Omega})$ and $\mathcal{K}_{\partial\Omega}:C^{1+\theta}(\partial\Omega)\to C^{2+\theta}(\partial\Omega)$ associated with (6.4) and (6.5), respectively, in the sense that for any given $\varphi_1\in C^\theta(\overline{\Omega})$ (resp. $\varphi_2\in C^{1+\theta}(\partial\Omega)$), $v=\mathcal{K}_\Omega\varphi_1$ (resp. $w=\mathcal{K}_{\partial\Omega}\varphi_2$) is the unique solution to (6.4) (resp. (6.5)). In fact, $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega}$ are two bijections and homeomorphisms [25]. The result of Amann [3] shows that $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega}$ can be extended to continuous linear maps from $C(\overline{\Omega})$ to $W_l^2(\Omega)$ and from $C(\partial\Omega)$ to $W_l^1(\Omega)$ for any $1<l<\infty$, respectively. It then follows from the strong maximum principle and Hopf's lemma that $\mathcal{K}_\Omega$ and $\mathcal{K}_{\partial\Omega}$ are both strongly positive in the sense that $\mathcal{K}_\Omega\varphi_1 \in \mathring{\mathbb{X}}$ for any $\varphi_1 \in \mathbb{X}\setminus\{0\}$, while $\mathcal{K}_{\partial\Omega}\varphi_2 \in \mathring{\mathbb{X}}$ for any nonnegative and nontrivial $\varphi_2 \in C(\partial\Omega)$. Consider the operator $\mathcal{A}(\lambda)v := \mathcal{K}_\Omega((M_1 + \lambda e^{\alpha P(x)}k(x))v) + \mathcal{K}_{\partial\Omega}(M_2 v)$ and choose a constant $M_1 > 0$ large enough such that $M_1 + \lambda_1 e^{\alpha P(x)}k(x) > 0$ on $\overline{\Omega}$, then $\mathcal{A}(\lambda_1)$ is also strongly positive. Note that if a nonnegative function $u \in C(\overline{\Omega})$ solves the equation

$$u - \mathcal{A}(\lambda)u - \lambda[\mathcal{K}_\Omega(e^{\alpha P(x)}k(x)u^p) + \mathcal{K}_{\partial\Omega}(r(x)u^q)] = 0, \tag{6.6}$$

then u is a solution of (6.2). Set $\mathcal{L}(\lambda)v := v - \mathcal{A}(\lambda)v$ and $\mathcal{B}(\lambda, v) = -\lambda[\mathcal{K}_\Omega(e^{\alpha P(x)}k(x)v^p) + \mathcal{K}_{\partial\Omega}(r(x)v^q)]$. By checking the conditions $N(\mathcal{L}(\lambda_1)) = \mathrm{span}\{\phi_1\}$, $\mathrm{codim}R(\mathcal{L}(\lambda_1)) = 1$ and $\mathcal{L}'(\lambda_1)[\phi_1] \notin R(\mathcal{L}(\lambda_1))$, we can still obtain the local bifurcating solution in part (i) from Crandall-Rabinowitz's Theorem [20].

Next, we prove the global bifurcation result in part (ii). Here we can adopt a global bifurcation theorem [54, Theorems 4.3 and 4.4] to the operator equation $\mathcal{L}(\lambda)u + B(\lambda, u) = 0$, and obtain the following three alternatives for the connected component $\mathscr{C}$:
(a) it is not compact;
(b) it meets another bifurcation point $(\hat{\lambda}, 0)$ with $\hat{\lambda} \neq \lambda_1$;
(c) it contains a point $(\lambda, \psi) \in \mathbb{R} \times R(\mathcal{L}(\lambda_1)) \setminus \{0\}$.

We can prove that the third case is impossible. We observe that $u \in \mathring{\mathbb{X}}$ if $u \in \mathbb{X}\setminus\{0\}$ satisfies (6.6). Suppose to the contrary that **case** (c) holds, then $\psi \in \mathring{\mathbb{X}}$ by the fact that $\psi \in \mathscr{C}$ and $\psi \neq 0$. Since $\psi \in R(\mathcal{L}(\lambda_1))$, there exists $v \in C(\overline{\Omega})$ such that $v - \mathcal{A}(\lambda_1)v = \psi$. By the positivity of ϕ_1, we can choose $\gamma > 0$ large enough so that $v_\gamma := v + \gamma\phi_1 \in \mathring{\mathbb{X}}$. Thus $v_\gamma - \mathcal{A}(\lambda_1)v_\gamma = \psi \in \mathring{\mathbb{X}}$, which combined with the strong positivity of $\mathcal{A}(\lambda_1)$ contradicts Theorem 3.2 in [4]. Hence **case** (c) is impossible. As for **case** (b), we can obtain from Lemma 4.2 (i) that $\hat{\lambda}$ must be the principal eigenvalue of (6.3), and then $\hat{\lambda} = 0$, which is impossible since there is no bifurcation near $(\lambda, u) = (0, 0)$ by $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x < 0$ and Theorem 4.7. Therefore, $\mathscr{C}$ is not compact and part (ii.a) is proved.

For part (ii.b), suppose to the contrary that there exists a sequence of positive solutions (λ_j, u_j) of (6.6) converging to $(\bar{\lambda}, 0) \in \mathbb{R}^+ \times C(\overline{\Omega})$ with $\bar{\lambda} \neq 0, \lambda_1$. Set $w_j = u_j/\|u_j\|_{C(\overline{\Omega})}$, then passing to a subsequence if necessary, there exists nonnegative and nontrivial w_* such that $w_j \to w_*$ as $j \to \infty$. Thus, $w_* - \mathcal{A}(\bar{\lambda})w_* = 0$, which means that $\bar{\lambda}$ is a principal eigenvalue of (6.3), a contradiction. This completes the proof. □

Secondly, we study the bifurcation from the line of trivial solution curve Γ_{u_1}.

Proposition 6.3. *Suppose that* **(H1)** *holds, and* $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x < 0$. *Set*

$$T(u) := (1 - u^{p-1})\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x + u^{q-1}\int_{\partial\Omega} e^{\alpha P(x)}r(x)\mathrm{d}S = 0. \tag{6.7}$$

If one of the following three conditions holds:

(a) $p > q > 1$; *or*
(b) $p = q > 1$ *and* $\int_\Omega e^{\alpha P(x)}k(x)\mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)}r(x)\mathrm{d}S$; *or*
(c) $q > p > 1$, *and* $\int_{\partial\Omega} e^{\alpha P(x)}r(x)\mathrm{d}S > 0$,

then (6.7) admits a zero $u_* > 0$ and $(0, u_*)$ is a bifurcation point with respect to Γ_{u_1}. That is, the solution set of (6.2) near $(0, u_*)$ consists exactly of the trivial solution curve Γ_{u_1} and the curve

$$\Sigma_1 = \{(\lambda_1(s), u_1(s)) : s \in (-\delta, \delta) \subset \mathbb{R}\},$$

where $\lambda_1(s)$ and $u_1(s) = u_* + \sigma_1(s) + \tilde{\eta}(\lambda_1(s), \sigma_1(s))$ are C^1 functions satisfying $\lambda_1(0) = \sigma_1(0) = 0$, $\lambda_1'(0) = 1$,

$$\sigma_1'(0) = -\frac{(1 - pu_*^{p-1}) \int_\Omega e^{\alpha P(x)} k(x) v_1 \mathrm{d}x + q u_*^{q-1} \int_{\partial\Omega} e^{\alpha P(x)} r(x) v_1 \mathrm{d}S}{(1 - pu_*^{p-1}) \int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x + q u_*^{q-1} \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S},$$

and v_1 is the unique solution of

$$\begin{cases} \nabla \cdot \left[e^{\alpha P(x)} \nabla v\right] + e^{\alpha P(x)} k(x)(u_* - u_*^p) = 0, & x \in \Omega, \\ \partial_{\vec{n}} v = r(x) u_*^q, & x \in \partial\Omega, \\ \int_\Omega v(x) \mathrm{d}x = 0. \end{cases}$$

Proof. It follows from Lemma 4.2 (ii) that a bifurcation point $(0, u_*)$ in Γ_{u_1} satisfies (6.7). Note that

$$T(u) = \int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x - u^{q-1} \Big(u^{p-q} \int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S \Big).$$

Obviously, $T(0) = \int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x < 0$ and $\lim_{u \to \infty} T(u) = +\infty$. This implies that there exists a positive constant u_* satisfying (6.7) and then the assumptions **(H2)** and **(H4)** in Theorem 4.6 hold. □

The above proposition only shows the existence of bifurcation points on the line of trivial solution Γ_{u_1}. In the case $p = 2$, we can further explore exact number of bifurcation points on Γ_{u_1}.

Proposition 6.4. *Suppose that* **(H1)** *holds, and $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x < 0$. Set $p = 2$.*

(a) For $1 < q < 2$, problem (6.2) has only one bifurcation point $(0, u_*)$ on Γ_{u_1}.
(b) For $q = 2$, if $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S$, then problem (6.2) has only one bifurcation point $(0, u_*)$ on Γ_{u_1}, otherwise, it has no bifurcation point on Γ_{u_1}.
(c) For $q > 2$, if $\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S \geq 0$, then problem (6.2) has only one bifurcation point $(0, u_*)$ on Γ_{u_1}; if either $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x \geq \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S$, or $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S < 0$ and $\hat{c} < \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, then problem (6.2) has no bifurcation point on Γ_{u_1}; if $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S < 0$ and $\hat{c} > \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, then problem (6.2) has exactly two bifurcation points $(0, u_{1*})$ and $(0, u_{2*})$ on Γ_{u_1}, where u_{1*} and u_{2*} are two zeros of (6.7) with $u_{1*} < u_{2*}$. Here $\hat{c} = \frac{\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x}{\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S}$.

Proof. Thanks to Lemma 4.2 (ii), we know that if $(0, u_*)$ with $u_* > 0$ is a bifurcation point of (1.6) with respect to Γ_{u_1}, then u_* satisfies (6.7). Notice that

$$T'(u) = (q-1)u^{q-2}\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S - \int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x,$$

then when $u_* > 0$ satisfies (6.7) and $T'(u_*) \neq 0$, $(0, u_*)$ is a bifurcation point of (1.6) on Γ_{u_1}. When $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S \geq 0$, $T'(u) > 0$ for all $u > 0$. In view of the fact that $T(0) < 0$ and $\lim_{u\to\infty} T(u) = +\infty$, we obtain a unique bifurcation point $(0, u_*)$ on Γ_{u_1}, where u_* is the unique positive zero of (6.7). In the following, we consider the situation $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0$.

Suppose that $1 < q < 2$ and $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0$. Then the equation $T'(u) = 0$ admits a unique zero $\hat{u}$ so that $T'(u) < 0$ in $(0, \hat{u})$ and $T'(u) > 0$ in $(\hat{u}, +\infty)$. Hence, we can obtain a unique bifurcation point $(0, u_*)$ on Γ_{u_1}.

Suppose that $q = 2$ and $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0$. If $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$, then $T'(u) > 0$ for all $u > 0$. Since $T(0) < 0$ and $\lim_{u\to\infty} T(u) = \infty$, we obtain a unique bifurcation point $(0, u_*)$ on Γ_{u_1}. If $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x \geq \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$, then $T(u) \leq \int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x < 0$, which implies the nonexistence of bifurcation point on Γ_{u_1}.

Suppose that $q > 2$ and $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0$. Then the equation $T'(u) = 0$ admits a unique zero $\hat{u} = \left(\frac{\hat{c}}{q-1}\right)^{\frac{1}{q-2}}$ so that $T'(u) > 0$ in $(0, \hat{u})$ and $T'(u) < 0$ in $(\hat{u}, +\infty)$. If $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x \geq \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$, then $\hat{c} \leq 1 < \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, and

$$\begin{aligned} T(\hat{u}) &= (1-\hat{u})\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x + \hat{u}^{q-1}\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S \\ &= \left[1 - \frac{q-2}{q-1}\left(\frac{\hat{c}}{q-1}\right)^{\frac{1}{q-2}}\right]\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x < 0, \end{aligned}$$

which implies that $T(u) < 0$ for all $u > 0$. Hence problem (6.2) admits no bifurcation point on Γ_{u_1}. In the case of $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0$ and $\hat{c} < \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, we can still obtain the nonexistence of bifurcation point on Γ_{u_1}. Finally, if $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x < \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S < 0$ and $\hat{c} > \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, then $T(\hat{u}) > 0$. Note that $T(0) < 0$ and $\lim_{u\to\infty} T(u) = -\infty$, we can exactly obtain two bifurcation points $(0, u_{1*})$ and $(0, u_{2*})$ on Γ_{u_1}. The proof is completed. □

From Theorem 4.7, we have the profile of solutions to (6.2) near $(0, 0)$.

Proposition 6.5. *Suppose that* **(H1)** *holds, and* $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x < 0$. *Then the nonnegative solution set of* (6.2) *near* $(0, 0)$ *is the union of* Γ_0 *and* Γ_{u_1}.

6.2. Case (2): $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x = 0$

When $\int_{\Omega} e^{\alpha P(x)} k(x)\mathrm{d}x = 0$, the conditions in Theorems 4.3 and 4.6 cannot hold. If $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S \neq 0$, then the condition **(H2)**$'$ in Theorem 4.5 is satisfied, which means there is no bifurcation point on Γ_{u_1}. In the following, we investigate the positive solution of (6.2) bifurcating from $(0, 0)$.

Proposition 6.6. *Suppose that* **(H1)** *holds, and* $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x = 0 > \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$. *When* $p \geq 2$ *and* $q = 2$, *the solution set of* (6.2) *near* $(0, 0)$ *consists exactly of the curves* Γ_0, Γ_{u_1} *and*

$$\Sigma_2 = \{(\lambda_2(s), u_2(s)) : s \in (-\delta, \delta) \subset \mathbb{R}\},$$

where $\lambda_2(s)$ *and* $u_2(s) = \sigma_2(s) + \tilde{\eta}(\lambda_2(s), \sigma_2(s))$ *are* C^1 *functions such that* $\lambda_2(0) = \sigma_2(0) = 0$, $\lambda_2'(0) = 1$, $\sigma_2'(0) = -\frac{\int_\Omega e^{\alpha P(x)}|\nabla \xi_1|^2 \mathrm{d}x}{\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S}$, *and* ξ_1 *is the unique solution of*

$$\begin{cases} -\nabla \cdot \left[e^{\alpha P(x)} \nabla \xi\right] = e^{\alpha P(x)} k(x), & x \in \Omega, \\ \partial_{\vec{n}} \xi = 0, & x \in \partial\Omega. \end{cases}$$

Proof. In this case, we see that $f(x, u) = k(x)(1 - u^{p-1})$ and $\beta(x, u) = r(x)u$. One can easily check that the conditions **(H3)**$''$ and **(H5)** hold. Thus, Theorem 4.8 can be applied to obtain the desired conclusions. □

Proposition 6.6 establishes the existence of a positive solution $(\lambda, u_{2\lambda})$ of (6.2) for $0 < \lambda \ll 1$, which satisfies $u_{2\lambda} \to 0$ in $C(\overline{\Omega})$ as $\lambda \searrow 0$. Next, we show that (6.2) with $p = q = 2$ has a second positive solution growing up to infinity as $\lambda \searrow 0$. Suppose that $u \in C^2(\overline{\Omega})$ is a nonnegative solution to (6.2) with $p = q = 2$. Let $u = u_{2\lambda} + v$, then v satisfies

$$\begin{cases} \nabla \cdot [e^{\alpha P(x)} \nabla v] + \lambda e^{\alpha P(x)} k(x)(v - 2u_{2\lambda} v - v^2), & x \in \Omega, \\ \partial_{\vec{n}} v = \lambda r(x)(2u_{2\lambda} v + v^2), & x \in \partial\Omega. \end{cases} \tag{6.8}$$

When $N = 2$ or 3, the Sobolev space $W_2^1(\Omega)$ is compactly embedded into $L^3(\Omega)$ and the usual trace operator $W_2^1(\Omega) \to L^3(\partial\Omega)$ is also compact. $(\lambda, v_\lambda) \in (0, +\infty) \times W_2^1(\Omega)$ is called a weak solution of (6.8) if

$$\int_\Omega e^{\alpha P(x)} \nabla v_\lambda \cdot \nabla \varphi \mathrm{d}x - \lambda \int_\Omega e^{\alpha P(x)} k(x)(1 - 2u_{2\lambda}) v_\lambda \varphi + \lambda \int_\Omega e^{\alpha P(x)} k(x)(v_\lambda)^2 \varphi \mathrm{d}x$$
$$- 2\lambda \int_{\partial\Omega} e^{\alpha P(x)} r(x) u_{2\lambda} v_\lambda \varphi \mathrm{d}S - \lambda \int_{\partial\Omega} e^{\alpha P(x)} r(x)(v_\lambda)^2 \varphi \mathrm{d}S = 0$$

for any $\varphi \in W_2^1(\Omega)$. It follows from the regularity theory of elliptic equation that a weak solution of (6.8) belongs to $C^2(\overline{\Omega})$, as the desired sense.

Now we can formulate a constrained minimization problem related to (6.8): for the function

$$J_\lambda(v) := \frac{\lambda}{3} \int_\Omega e^{\alpha P(x)} k(x)|v|^3 \mathrm{d}x - \frac{\lambda}{3} \int_{\partial\Omega} e^{\alpha P(x)} r(x)|v|^3 \mathrm{d}S, \quad \forall v \in M_\lambda,$$

where

$$M_\lambda = \Big\{ v \in W_2^1(\Omega) \Big| E_\lambda(v) := \frac{1}{2} \int_\Omega e^{\alpha P(x)} |\nabla v|^2 \mathrm{d}x - \frac{\lambda}{2} \int_\Omega e^{\alpha P(x)} k(x)(1 - 2u_{2\lambda}) v^2 - \lambda \int_{\partial\Omega} e^{\alpha P(x)} r(x) u_{2\lambda} v^2 \mathrm{d}x \le 1 \Big\},$$

seek for $v_\lambda \in M_\lambda$ such that

$$v_\lambda \neq 0 \quad \text{and} \quad J_\lambda(v_\lambda) = \inf_{v \in M_\lambda} J_\lambda(v). \tag{6.9}$$

To achieve this aim, we first prove that

$$\inf_{v \in M_\lambda} J_\lambda(v) < 0. \tag{6.10}$$

In fact, when $k : \Omega \to \mathbb{R}$ is sign-changing in Ω and $\int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S < 0$, there exists a suitable C^1-function $\tilde{v}$ defined on $\overline{\Omega}$, whose support is contained in a neighborhood of $x \in \Omega$ satisfying $k(x) < 0$, so that $J_\lambda(\tilde{v}) < 0$. Then $J_\lambda(\varepsilon\tilde{v}) < 0$ and $E_\lambda(\epsilon\tilde{v}) \le 1$ for sufficiently small $\varepsilon > 0$. This proves (6.10).

Proposition 6.7. *Let $N = 2$ or 3. Suppose that* **(H1)** *holds, $k : \Omega \to \mathbb{R}$ is sign-changing in Ω and $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x = 0 > \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S$. Then we can find a constant $\bar{\lambda} > 0$ satisfying that for $\lambda \in (0, \bar{\lambda}]$, there exist some constant $C(\lambda) > 0$ and a minimizing sequence for (6.9), i.e., $J_\lambda(v_j) \searrow \inf_{v \in M_\lambda} J_\lambda(v) \in [-\infty, 0)$, such that $\|v_j\|_{W_2^1(\Omega)} \le C(\lambda)$.*

Proof. Decompose the space $W_2^1(\Omega) = \mathbb{R} \oplus W$, in which $W = \{\xi \in W_2^1(\Omega) | \int_\Omega \xi \mathrm{d}x = 0\}$ is equipped with a reduced norm $\|\cdot\|_W$ in $W_2^1(\Omega)$. Clearly, for $v = c + \xi \in W_2^1(\Omega)$ with $c \in \mathbb{R}$ and $\xi \in W$, $\|v\|^2_{W_2^1(\Omega)}$ and $|c|^2 + \|\xi\|_W^2$ are equivalent. One can also verify that for $\xi \in W$, $\|\xi\|_W^2$ is equivalent to $\int_\Omega |\nabla\xi|^2 \mathrm{d}x$. In addition, since $W \subset L^2(\Omega), L^2(\partial\Omega)$ are both continuous, we can find constants $C_0 > 0$ and λ^* satisfying that $C_0\|\xi\|_W^2 \le E_\lambda(\xi)$ for all $\xi \in W$ and $\lambda \in (0, \lambda^*]$. Based on this fact, a similar manner as that in [58, Proposition 5.2] can be carried out to prove that for any $\{v_j\} \in W_2^1(\Omega)$ satisfying that $E_\lambda(v_j) \le 1$ and $\|v_j\|_{W_2^1(\Omega)} \to +\infty$ as $j \to +\infty$, where $v_j = c_j + \xi_j \in \mathbb{R} \oplus W$, there exists a constant $C_1(\lambda) > 0$ such that $\limsup\limits_{j \to +\infty} \left\| \frac{\xi_j}{c_j} \right\|_W \le C_1(\lambda)$ and $C_1(\lambda) \to 0$ as $\lambda \searrow 0$. Since $W \subset L^3(\Omega), L^3(\partial\Omega)$ are both continuous and $\int_\Omega e^{\alpha P(x)} k(x) \mathrm{d}x > \int_{\partial\Omega} e^{\alpha P(x)} r(x) \mathrm{d}S$, there exist constants $\varepsilon_0 > 0$, $C_2 > 0$ and $\bar{\lambda} \in (0, \lambda^*)$ so that

$$C_1(\lambda) < \frac{\varepsilon_0}{2} \quad \text{for } \lambda \in (0, \bar{\lambda}], \tag{6.11}$$

and

$$\int_\Omega e^{\alpha P(x)} k(x) |1 + \xi|^3 \mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)} r(x) |1 + \xi|^3 \mathrm{d}S \ge C_2 \quad \text{for } \xi \in W \text{ and } \|\xi\|_W \le \varepsilon_0. \tag{6.12}$$

Now, we complete the proof. On the contrary, suppose that there is $\lambda \in (0, \bar{\lambda}]$ such that a minimizing sequence $\{v_j\} \subset M_\lambda$ contains a subsequence, still denoted by $\{v_j\}$, satisfying $\|v_j\|_{W_2^1(\Omega)} \to +\infty$ as $j \to +\infty$. Since $E_\lambda(v_j) \le 1$, for the subsequence $\{v_j\}$, it follows from (6.11) that $\limsup_{j\to+\infty} \left\| \frac{\xi_j}{c_j} \right\|_W \le \frac{\varepsilon_0}{2}$, where $v_j = c_j + \xi_j \in \mathbb{R} \oplus W$. Hence, by (6.12), we can find $j_0(\lambda) \ge 1$ so that for any $j \ge j_0(\lambda)$,

$$J_\lambda(v_j) = \frac{\lambda |c_j|^3}{3} \int_\Omega e^{\alpha P(x)} k(x) \left| 1 + \frac{\xi_j}{c_j} \right|^3 \mathrm{d}x - \frac{\lambda |c_j|^3}{3} \int_{\partial\Omega} e^{\alpha P(x)} r(x) \left| 1 + \frac{\xi_j}{c_j} \right|^3 \mathrm{d}S \ge \frac{\lambda |c_j|^3}{3} C_2,$$

a contradiction with (6.10) as $j \to +\infty$. The proof is completed. □

Consequently, we can obtain the existence of a second positive solution growing up to infinity as $\lambda \searrow 0$.

Proposition 6.8. *Let $N = 2$ or 3. Suppose that* **(H1)** *holds, $k : \Omega \to \mathbb{R}$ is sign-changing in Ω and $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x = 0 > \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$. Then Eq. (6.2) with $p = q = 2$ admits a positive solution (λ, u_λ) for $0 < \lambda \ll 1$, which satisfies that $u_\lambda > u_{2\lambda}$. In addition, $u_\lambda \to +\infty$ in $C(\overline{\Omega})$ as $\lambda \searrow 0$.*

Proof. By Proposition 6.7, the standard compactness argument can be carried out to show that there are a subsequence of $\{v_j\}$, still denoted by $\{v_j\}$, and a function $v_\lambda \in W_2^1(\Omega)$ for $\lambda \in (0, \bar{\lambda}]$ such that $v_j \to v_\lambda$ weakly in $W_2^1(\Omega)$ and $v_j \to v_\lambda$ strongly in both $L^3(\Omega)$ and $L^3(\partial\Omega)$. Then taking $j \to +\infty$ yields that

$$J_\lambda(v_j) \to \frac{\lambda}{3} \int_\Omega e^{\alpha P(x)} k(x) |v_\lambda|^3 \mathrm{d}x - \frac{\lambda}{3} \int_{\partial\Omega} e^{\alpha P(x)} r(x) |v_\lambda|^3 \mathrm{d}S = J_\lambda(v_\lambda) = \inf_{v \in M_\lambda} J_\lambda(v) > -\infty.$$

It follows from the lower semi-continuity of $E_\lambda(\cdot)$ that $E_\lambda(v_\lambda) \le 1$. Here, the minimizer v_λ can be chosen as a nonnegative function (if not, then v_λ can be replaced by $|v_\lambda|$).

In what follows, we show that $E_\lambda(v_\lambda) = 1$. By way of contradiction, suppose that $E_\lambda(v_\lambda) < 1$, which means v_λ is an interior point in M_λ. Then for any $\varphi \in W_2^1(\Omega)$, $v_\lambda + s\varphi \in M_\lambda$ ($|s| \ll 1$) and

$$\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda(v_\lambda + s\varphi)|_{s=0} = 0. \tag{6.13}$$

A straightforward calculation gives that

$$\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda(v_\lambda + s\varphi)|_{s=0} = \lambda \left(\int_\Omega e^{\alpha P(x)} k(x) (v_\lambda)^2 \varphi \mathrm{d}x - \int_{\partial\Omega} e^{\alpha P(x)} r(x) (v_\lambda)^2 \varphi \mathrm{d}S \right), \quad \forall \varphi \in W_2^1(\Omega).$$

When $\varphi = v_\lambda$, we have $\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda(v_\lambda + s\varphi)|_{s=0} = 3J_\lambda(v_\lambda) < 0$, contradicting with (6.13). Hence, $E_\lambda(v_\lambda) = 1$. Moreover, $v_\lambda \ge, \not\equiv 0$ and $J_\lambda(v_\lambda) = \inf_{v \in M_\lambda} J_\lambda(v) < 0$.

In the following, it suffices to show that for any $\varphi \in W_2^1(\Omega)$, the minimizer v_λ satisfies $J_\lambda'(v_\lambda)\varphi + KE_\lambda'(v_\lambda)\varphi = 0$, in which K is the corresponding Lagrange multiplier. Taking $\varphi = v_\lambda$ gives that $K = -3J_\lambda(v_\lambda)/2$. Let $V_\lambda = v_\lambda/K$, then (λ, V_λ) is a weak solution of (6.8), which is nonnegative and nontrivial. Moreover, the strong maximum principle and Hopf's Lemma imply that $V_\lambda > 0$ on $\overline{\Omega}$. Therefore, we obtain a second positive solution $u_\lambda = u_{2\lambda} + V_\lambda$ of (6.2) as desired.

Finally, we prove that $u_\lambda \to +\infty$ in $C(\overline{\Omega})$ as $\lambda \searrow 0$. On the contrary, suppose that $\lambda_j \searrow 0$ and $\|u_{\lambda_j}\|_{C(\overline{\Omega})} \le C_3$ for some constant $C_3 > 0$ as $j \to +\infty$. It then follows from Amann's L^p regularity theory [3, Proposition 3.3] that $\|u_{\lambda_j}\|_{W_l^1(\Omega)}$ is bounded in $j \ge 1$ for any $1 < l < +\infty$. By the standard L^p theory and the Arzela-Ascoli theorem, we find a subsequence of $\{(\lambda_j, u_{\lambda_j})\}$, still denoted by $\{(\lambda_j, u_{\lambda_j})\}$, and a function $\bar{u} \in C^2(\overline{\Omega})$ such that $\lambda_j \searrow 0$ and $u_{\lambda_j} \to \bar{u}$ in $C^2(\overline{\Omega})$, where $\bar{u}$ is a nonnegative constant. However, Proposition 6.6 and Theorem 4.5 imply that there is a unique bifurcation curve near $(0, 0)$ and (6.2) has no bifurcation point on Γ_{u_1}, a contradiction. Consequently, we obtain that $u_\lambda \to +\infty$ in $C(\overline{\Omega})$ as $\lambda \searrow 0$. □

6.3. *Case (3): $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x > 0$*

When $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x > 0$, the condition **(H3)** in Theorem 4.3 is not satisfied, but the condition **(H3)′** in Theorem 4.7 holds. Hence, the nonnegative solution set of (6.2) near $(0, 0)$ is the union of Γ_0 and Γ_{u_1}. The following proposition shows the exact number of bifurcation points with respect to Γ_{u_1}, which can be proved by a similar argument as Proposition 6.4.

Proposition 6.9. *Suppose that* **(H1)** *holds, and $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x > 0$. Set $p = 2$.*

(a) For $1 < q < 2$, problem (6.2) has only one bifurcation point $(0, u_*)$ on Γ_{u_1}.
(b) For $q = 2$, if $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x > \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$, then problem (6.2) has only one bifurcation point $(0, u_*)$ on Γ_{u_1}, otherwise, there is no bifurcation point on Γ_{u_1}.
(c) For $q > 2$, if $\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S \le 0$, then problem (6.2) has only one bifurcation point $(0, u_*)$ on Γ_{u_1}; if either $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x \le \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S$, or $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x > \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S > 0$ and $\hat{c} < \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, then problem (6.2) has no bifurcation point on Γ_{u_1}; if $\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x > \int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S > 0$ and $\hat{c} > \frac{(q-1)^{q-1}}{(q-2)^{q-2}}$, then problem (6.2) has exactly two bifurcation points $(0, u_{1*})$ and $(0, u_{2*})$ on Γ_{u_1}, where u_{1*} and u_{2*} are two zeros of (6.7) with $u_{1*} < u_{2*}$. Here $\hat{c} = \frac{\int_\Omega e^{\alpha P(x)} k(x)\mathrm{d}x}{\int_{\partial\Omega} e^{\alpha P(x)} r(x)\mathrm{d}S}$.

7. Summary and discussion

In the theory of differential equations, steady-state bifurcation refers to the phenomenon that the non-trivial steady-state solution of the considered system goes from non-existence to existence or from existence to non-existence near the steady-state solution when the parameters change, which is often related to a breaking of symmetry [32]. One important tool for solving such problem is the Lyapunov-Schmidt reduction which reduces the original problem to a finite-dimensional one. A wide used steady-state bifurcation theorem is established by Crandall and Rabinowitz [20,21], which deals with the case for simple eigenvalue. Such a result is very convenient, since in applications, only some linearized operators need to be checked, and the Lyapunov-Schmidt reduction is not required. Recently, Liu et al. [39,40] formulated two symmetry-breaking bifurcation results in two kinds of degenerate cases, which are the complement of the work of Crandall and Rabinowitz [20,21]. Lyapunov-Schmidt reduction method is the main tool used in the work of Liu et al. [39,40].

In this paper, we study a general reaction-diffusion-advection single population model, where the boundary condition is assumed to be nonlinearly dependent on the population density, more complex than some existing work where the population is supposed to undergo the homogeneous no-flux boundary condition, or homogeneous Dirichlet boundary condition, see, e.g., [9,12,13,30,34]. Firstly, we prove the existence of the principal eigenvalue of an eigenvalue problem with indefinite weighted function. Then in Section 3, in order to study the stability of the trivial steady state $u = 0$ to system (1.5), we establish the relationship of the stability of any nonnegative steady state $u = \hat{u}$ to (1.5) and the sign the principal eigenvalue $\mu_1(\lambda, \hat{u})$ of the linearized eigenvalue problem at $u = \hat{u}$. In Section 4, we provide three types of bifurcation results for system (1.5), which show the existence of the nonconstant positive steady states:

(i) The first type of bifurcation result is obtained by the Crandall-Rabinowitz bifurcation theorem, which shows that when **(H1)** and **(H3)** are satisfied, there exists a critical value λ_1 such that a nonconstant positive steady state $u_{0\lambda}$ will bifurcate from Γ_0 for $|\lambda - \lambda_1| \ll 1$ (see Theorem 4.3);
(ii) The second type of bifurcation result is performed by the Lyapunov-Schmidt reduction, which shows that under the assumptions **(H1)**, **(H2)** and **(H4)**, system (1.5) admits a nonconstant positive steady state $u_{1\lambda}$ bifurcating from Γ_{u_1} for $0 < \lambda \ll 1$ (see Theorem 4.6);
(iii) The third type of bifurcation result concerning the degenerate simple eigenvalue is also derived by the Lyapunov-Schmidt reduction, which shows that under the assumptions **(H1)**, **(H3)**$''$ and **(H5)**, system (1.5) admits a nonconstant steady state $u_{2\lambda}$ bifurcating from $(0, 0)$ for $0 < \lambda \ll 1$ (see Theorem 4.8).

Moreover, the stability of the above bifurcating steady state is provided by calculating the sign of the associated principal eigenvalue. We point out here that the second type of bifurcation (Theorem 4.6) and the third type of bifurcation (Theorem 4.8) can also be obtained by employing the crossing curve bifurcation result in [39, Theorem 2.1] and the bifurcation result from a degenerate simple eigenvalue in [40, Theorem 2.3], respectively. In this paper, we only use the Lyapunov-Schmidt reduction method to derive the second and third bifurcations, in order to better understand the conditions under which the second and third bifurcations appear, under which conditions they cannot appear, and how these two bifurcations appear.

As applications of our main result, we also investigate two special cases of system (1.5). For the case of Eq. (5.1), where the species is supposed to adopt zero interior growth and nonlinear boundary reaction of monostable type, we present a complete global dynamics when the coefficient of linear term in the boundary condition is sign-changing; see Theorems 5.6 and 5.13. Especially, we use the Lyapunov-Schmidt reduction method to obtain the fourth type of the steady state bifurcation from the trivial solution curve Γ_{u_1}; see Lemma 5.2. As for the case of Eq. (6.1), where the species is supposed to adopt sublinear growth and superlinear boundary condition, we show the stability of trivial steady state $u = 0$ and provide some sufficient conditions on the existence of three kinds of bifurcating positive steady states. Furthermore, we extend the first local bifurcation for (6.1) to be a global one, show the exact number of bifurcation points for the second local bifurcation for (6.1) with $p = 2$ (logistic interior growth case), and establish the existence of an additional positive solution occurring near the third type of the bifurcating positive steady state; see Propositions 6.2, 6.4, 6.8 and 6.9.

Now, we give some conclusions for our original model (1.1), which has only one trivial steady state $u = 0$ for all $d > 0$ and $a > 0$. Recall the transformation in Section 1 and set the following conditions:

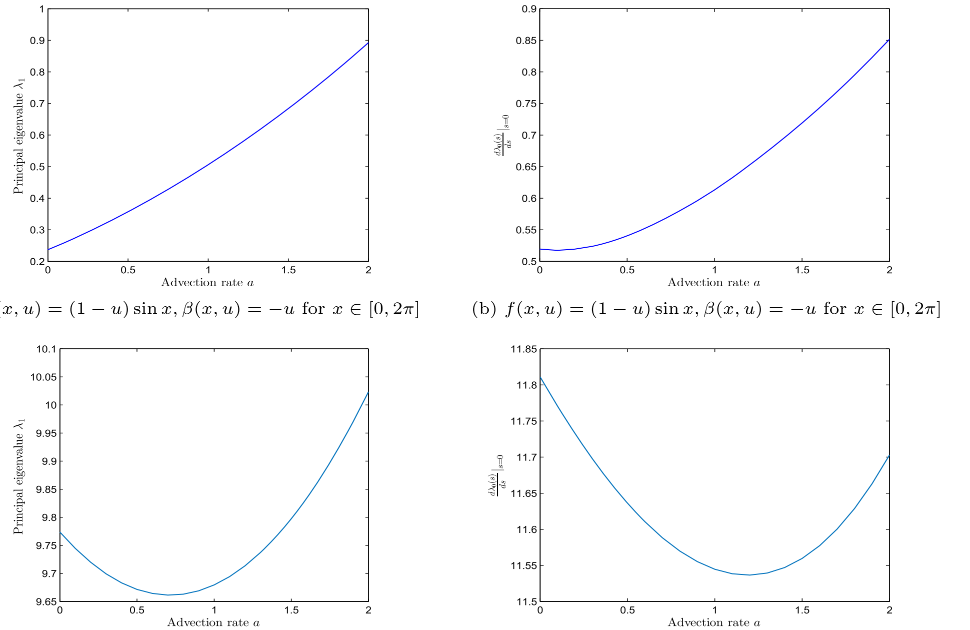
(C1) $\int_\Omega e^{\frac{a}{d}P(x)} F(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{a}{d}P(x)} B(x,0)\mathrm{d}S < 0$;
(C1)$'$ $\int_\Omega e^{\frac{a}{d}P(x)} F(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{a}{d}P(x)} B(x,0)\mathrm{d}S > 0$;
(C1)$''$ $\int_\Omega e^{\frac{a}{d}P(x)} F(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{a}{d}P(x)} B(x,0)\mathrm{d}S = 0$;
(C2) $\int_\Omega e^{\frac{2a}{d}P(x)} F_u(x,0)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{2a}{d}P(x)} B_u(x,0)\mathrm{d}S \neq 0$;
(C3) $\int_\Omega e^{\frac{a}{d}P(x)} F(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{a}{d}P(x)} B(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}S \neq 0$ for any $u_* > 0$;
(C3)$'$ $\int_\Omega e^{\frac{a}{d}P(x)} F(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{a}{d}P(x)} B(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}S = 0$ for some $u_* > 0$;
(C4) $\int_\Omega e^{\frac{2a}{d}P(x)} F_u(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{2a}{d}P(x)} B_u(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}S \neq 0$ for some $u_* > 0$.

Then we can conclude that, under the assumption **(H1)**,

(i) *if $F(x,0)$ is sign-changing in Ω, $B(x,0)$ is sign-changing on $\partial\Omega$, and **(C1)** holds, then there exists a critical value $d_1 = \frac{1}{\lambda_1} > 0$ such that the trivial steady state $u = 0$ of (1.1) is locally asymptotically stable for $d > d_1$, while unstable for $0 < d < d_1$. Moreover, a positive steady state $u_{0d} = e^{aP(x)/d}\tilde{u}_{0\lambda}$ of (1.1) will bifurcate from $u = 0$ near $d = d_1$, and there is no bifurcation occurring at $u = 0$ for sufficiently large $d > d_1$ (see Theorems 3.4 (i), 4.3 and 4.7);*
(ii) *if $F(x,0)$ is sign-changing in Ω, $B(x,0)$ is sign-changing on $\partial\Omega$, and **(C1)**$'$ holds, then the trivial steady state $u = 0$ of (1.1) is unstable for all $d > 0$, and (1.1) admits no nontrivial steady state near $u = 0$ for sufficiently large $d > 0$ (see Theorems 3.4 (ii), 4.7);*
(iii) *if $F(x,0)$ is sign-changing in Ω, $B(x,0)$ is sign-changing on $\partial\Omega$, and **(C1)**$''$ holds, then the trivial steady state $u = 0$ of (1.1) is unstable for all $d > 0$. Furthermore, if **(C2)** is satisfied, then a positive steady state $u_{2d} = e^{aP(x)/d}\tilde{u}_{2\lambda}$ will bifurcate from $u = 0$ for sufficiently large $d > 0$, which is stable (see Theorems 3.4 (ii) and 4.8);*
(iv) *if **(C3)** holds, then there is no bifurcation occurring at $e^{aP(x)/d}u_*$ for any constant $u_* > 0$ and sufficiently large $d > 0$ (see Theorem 4.5);*
(v) *if **(C3)**$'$ and **(C4)** hold, then a positive steady state $u_{1d} = e^{aP(x)/d}\tilde{u}_{1\lambda}$ will bifurcate from $e^{aP(x)/d}u_*$ for sufficiently large $d > 0$, which is stable when $\int_\Omega e^{\frac{2a}{d}P(x)} F_u(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{2a}{d}P(x)} B_u(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}S < 0$, while unstable when $\int_\Omega e^{\frac{2a}{d}P(x)} F_u(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}x + \int_{\partial\Omega} e^{\frac{2a}{d}P(x)} B_u(x, e^{\frac{a}{d}P(x)}u_*)\mathrm{d}S > 0$ (see Theorem 4.6).*

7.1. The effects of advection rate

From Section 4.1, we see that the first type of the positive bifurcating steady state occurs at the critical value λ_1, which is the unique positive principal eigenvalue of (4.3), and the sign of $\lambda_0'(0)$ determines the bifurcation direction. It was found in [9, Section 3] and [42,67] that the positive principal eigenvalue of the linear diffusion-advection operator with the no-flux boundary condition may not be monotone with respect to the advection rate, and the theoretical result about the monotonicity with respect to the advection rate is still unsolved. It is also difficult to discuss the dependence of λ_1 and $\lambda_0'(0)$ on the advection rate α (or a in model (1.1)). Here, we use the finite element and finite difference methods [65] to approximate the principal eigenvalue of (4.3) and the value $\lambda_0'(0)$ in two situations: $P(x) = f(x,0) = F(x,0)$ and $P(x) = x$.



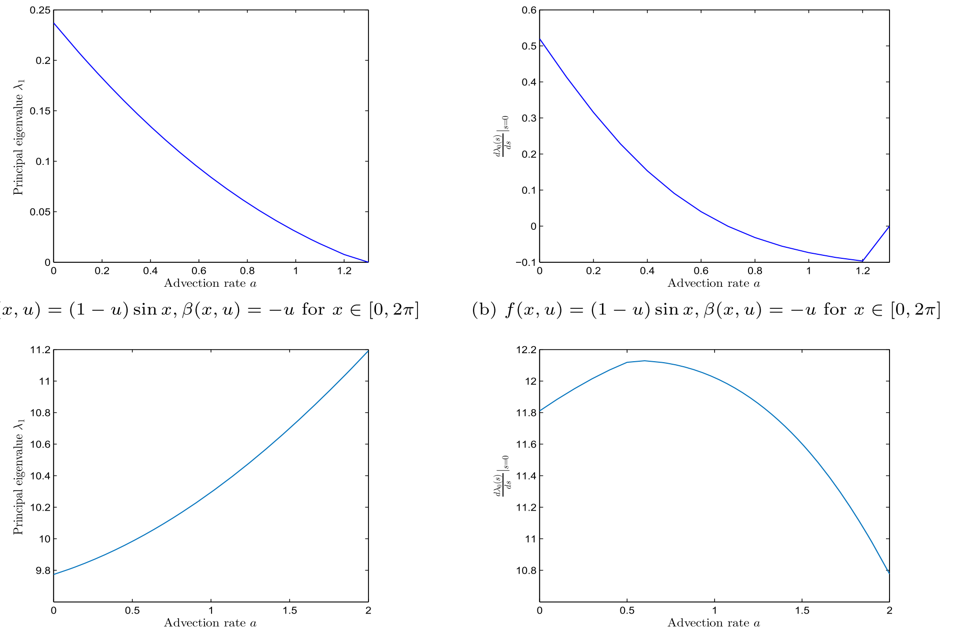
(a) $f(x,u) = (1-u)\sin x, \beta(x,u) = -u$ for $x \in [0, 2\pi]$ (b) $f(x,u) = (1-u)\sin x, \beta(x,u) = -u$ for $x \in [0, 2\pi]$

(c) $f(x,u) = (1-u)(1-x), \beta(x,u) = -u$ for $x \in [0, 2\pi]$ (d) $f(x,u) = (1-u)(1-x), \beta(x,u) = -u$ for $x \in [0, 2\pi]$.

Fig. 1. The effects of advection rate a on the critical value λ_1 and the value $\lambda_0'(0)$ determining bifurcation direction. Here, the diffusion rate $d = 2$, and the function determining the advection direction is $P(x) = x$, which means the species lives in the river environment.

Choose the logistic interior growth $f(x,u)u = k(x)[1-u]u$ and the nonlinear boundary function $\beta(x,u)u = -u^2$. In the situation $P(x) = x$, which means that the species lives in the river environment, Fig. 1 shows that for different resource functions $k(x) = \sin x$ or $k(x) = 1 - x$, the principal eigenvalue λ_1 and the value $\lambda_0'(0)$ may be monotone or nonmonotone with respect to the advection rate a. In the situation $P(x) = f(x,0) = F(x,0)$, which represents the advection direction along the gradient of the resource, Fig. 2 also shows that for different resource functions, the dependence of the principal eigenvalue λ_1 and the value $\lambda_0'(0)$ on the advection rate a is different, and the advection rate can even affect the bifurcation direction. As for the second and the third types of the positive bifurcating steady states, it can be seen in Theorems 4.6 and 4.8, respectively, that $\lambda_1'(0) = \lambda_2'(0) = 1$, which means that these two types of bifurcations occur at the right side of $\lambda = 0$. Hence, the advection rate cannot affect the bifurcation directions of the second and the third types of steady state bifurcations.

Next, we display the influence of the advection rate on the density distribution of the species u. In river environment, i.e., $P(x) = x$ for $x \in (0, 2\pi)$, Fig. 3 (a) and (b) show that when the advection rate is not too large ($a = 0.2$ or $a = 1$), the density of the species u converges to a positive steady state, and such steady state concentrates more on the downstream end $x = 2\pi$ of the river as the advection rate a increases. However, Fig. 3 (c) shows that when the advection rate a is large ($a = 2$), the species becomes extinct. These numerical results are consistent with the "drift paradox" in the literature [5,28]. When the species undergoes the directed movement along the gradient of the resource, i.e., $P(x) = F(x,0)$, Fig. 4 shows that the steady state concentrates more on the maximum point of resource $F(x,0) = \sin x$ as the advection rate a increases.



(a) $f(x,u) = (1-u)\sin x, \beta(x,u) = -u$ for $x \in [0, 2\pi]$ (b) $f(x,u) = (1-u)\sin x, \beta(x,u) = -u$ for $x \in [0, 2\pi]$

(c) $f(x,u) = (1-u)(1-x), \beta(x,u) = -u$ for $x \in [0, 2\pi]$ (d) $f(x,u) = (1-u)(1-x), \beta(x,u) = -u$ for $x \in [0, 2\pi]$.

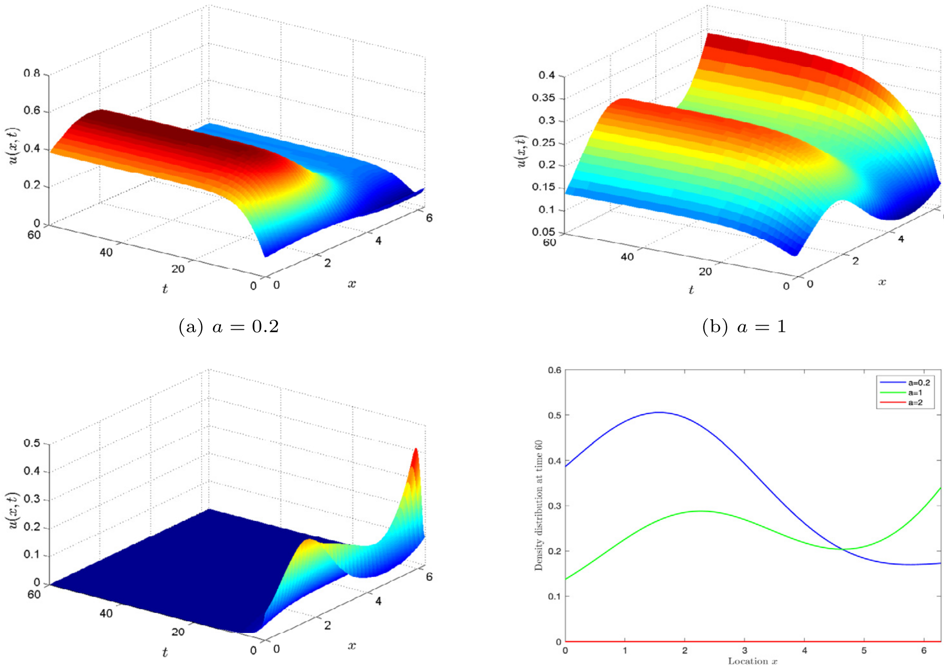
Fig. 2. The effects of advection rate a on the critical value λ_1 and the value $\lambda_0'(0)$ determining bifurcation direction. Here, the diffusion rate $d = 2$, and the function determining the advection direction is $P(x) = f(x,0) = F(x,0)$, which represents the directed movement along the gradient of the resource.

Meanwhile, Fig. 3 together with Fig. 4 shows that advection direction can also affect the density distribution of the species.

7.2. The effects of nonlinear boundary condition

Section 5 exhibits a complete global dynamics for the reaction-diffusion-advection model, which admits zero interior growth and monostable nonlinear boundary condition. For Eq. (5.1), Lemma 5.2 and Theorem 5.11 establish the existence of a positive steady state bifurcating from Γ_{u_1} and Γ_0, respectively, under different assumptions. Moreover, Theorems 5.6 and 5.13 show that the solution of (5.1) converges to a nonconstant positive steady state. The above results for the dynamics of (5.1) are quite similar to that of the model with interior growth of monostable type and no-flux boundary condition (see [10,24,35]). Clearly, the nonconstant positive steady state of (5.1) appears due to nonlinear boundary conditions, but the nonconstant positive steady state of the model considered in [10,24,35] appears due to nonlinear interior reaction.

Models (1.2) and (1.3) only contain logistic interior growth and linear boundary condition. It is shown in [12, Section 3] that the global dynamics of such scalar parabolic problem is only determined by the sign of the principal eigenvalue of the linearized operator at zero. Section 6 shows that the dynamics of the reaction-diffusion-advection model with logistic interior growth and nonlinear boundary condition is more complex. Proposition 6.2 establishes the existence of a global parameterized bifurcating positive steady state $u_{0\lambda}$ from $(\lambda_1, 0)$. Such result for the first type of the bifurcating solution is classical. One can refer to [12, Section 3] for some similar results for the model with linear boundary condition. For the second type of the bifurcating steady



(a) $a = 0.2$ (b) $a = 1$

(c) $a = 2$ (d) Density distribution in time 60.

Fig. 3. The effects of advection rate a on the density distribution of the species which lives in the river environment. Here, $d = 2$, $P(x) = x$, $F(x,u) = (1-u)\sin x$, $B(x,u) = -e^{-ax/d}u$ for $x \in (0, 2\pi)$, and the initial data is $u_0(x) = 0.1e^{\frac{0.1}{d}\sin\frac{x}{2}} \cdot e^{\frac{a}{d}\sin x}$ for $x \in (0, 2\pi)$.

state $u_{1\lambda}$, Propositions 6.4 and 6.9 show that model (6.1) can even admit exact two bifurcation points on Γ_{u_1}. As for the third type of the bifurcating steady state $u_{2\lambda}$, Proposition 6.8 shows the existence of the other steady state for $0 < \lambda \ll 1$, which is large than $u_{2\lambda}$ and will grow up to infinity as $\lambda \searrow 0$. Such multiplicity and growing-up property can not occur for the model with logistic interior growth and linear boundary condition.

At the end of this paper, we remark that in Theorem 4.8, the assumption **(H3)**$''$ is a degenerate condition for the function $G(\lambda, \sigma)$, and the assumption **(H5)** serves as a non-degenerate condition for the function $H(\lambda, \sigma)$. However, when **(H5)** is invalid, a second degenerate case happens for the function $H(\lambda, \sigma)$. Then one can still define function related to H and adopt Lemma 2.5 in [39]. In that case, a second degenerate bifurcation will occur at some points $(0, u_*)$ with $u_* > 0$.

Data availability

No data was used for the research described in the article.

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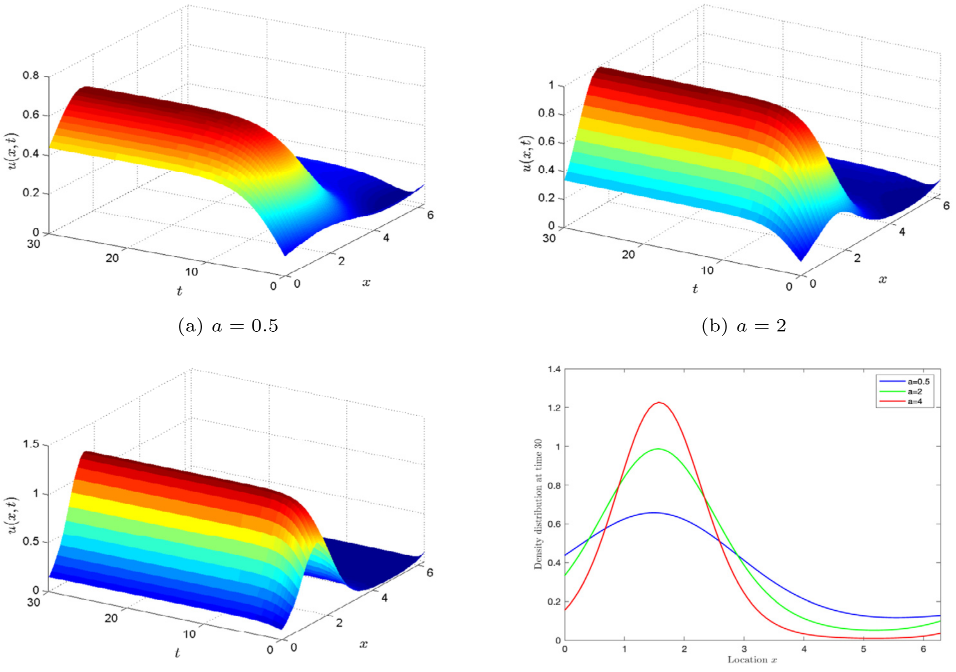

(a) $a = 0.5$　(b) $a = 2$

(c) $a = 4$　(d) Density distribution in time 30.

Fig. 4. The effects of advection rate a on the density distribution of the species with the advection direction along the gradient of the resource. Here, $d = 2$, $P(x) = \sin x$, $F(x, u) = (1 - u)\sin x$, $B(x, u) = -u$ for $x \in (0, 2\pi)$, and the initial data is $u_0(x) = 0.1e^{\frac{0.1}{d}\sin\frac{x}{2}} \cdot e^{\frac{a}{d}\sin x}$ for $x \in (0, 2\pi)$.

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