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Stabilization role of inhibitory self-connections in a delayed neural network[☆]

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Abstract

In a delayed Hopfield neural network that is strongly connected with *non-inhibitory interconnections*, fast and inhibitory self-connections lead to global convergence to a unique equilibrium of the network. By applying monotone dynamical systems theory and an embedding technique, we prove that this conclusion remains true without the requirement of strong connectivity or non-inhibitory interconnections. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider the system of delay differential equations

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t-\tau_{ij})) + J_i, \quad i = 1, 2, \dots, n, \tag{1.1}$$

where a_{ij}, τ_{ij} and J_i are given constants and $\tau_{ij} \geq 0$. This is obtained from the original Hopfield artificial neural network model (see [12]) by appropriate rescaling under the assumption of identical neurons with equal input resistance and capacitance (see [3]). The time delays $\tau_{ij} \geq 0$ account for the finite switching speeds in the circuit of neurons (amplifiers). Usually the $n \times n$ matrix $A = (a_{ij})$ is assumed to be irreducible, which means that the network is strongly connected. The activation (or transfer) functions $g_j \in C^1$ are sigmoidal, strictly increasing, odd, and have $\lim_{u\to\pm\infty} g_j(u) = \pm 1$ and $g_j'(0) > g_j'(u)$. Since $a_{ij} g_j(u) = [a_{ij} g_j'(0)] g_j(u)/g_j'(0)$, reassigning a_{ij} allows us to assume $g_j'(0) = 1$. A typical such activation function is $g_j(u) = \tanh(u)$, which has been widely adopted for

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neural networks. For convenience, we itemize these assumptions on g_j as follows:

$$(H1)\quad g_j \in C^1(R), \qquad g'_j(u) > 0, \qquad \sup_{u\in R} g'_j(u) = g'_j(0) = 1.$$

$$(H2)\quad g_j(0) = 0, \qquad g_j(u) \text{ saturates at } \pm 1, \text{ i.e., } \lim_{u\to\pm\infty} g_j(u) = \pm 1.$$

Model (1.1) with $\tau_{ij} = 0$ has been widely used for several purposes; for example, content addressable memories, signal processing, parallel computations and optimizations; see, for example [12,17,21], and the references therein. When a neural network is designed for optimization problems, it is required that given an input, there be a well-defined computable solution for all possible initial states. From a mathematical point of view, this requires that the network should have a unique equilibrium that is globally attractive. Indeed, earlier applications of the Hopfield model with $\tau_{ij} = 0$ to optimization problems have suffered from the existence of a complicated set of equilibria (see [17]). The addition of time delay leads to even richer and more complicated dynamics such as sustained oscillations and various connecting orbits between equilibria and periodic orbits (see [3,4,6–8,13–15,19,20,23], and the references therein). Consequently, the global convergence of (1.1) with positive delay τ_{ij} is of great importance in applications. For the single delay case ($\tau_{ij} = \tau$), Bélair [2] and Ye et al. [24] obtained some sufficient conditions for global convergence of the network by using Lyapunov functions/functionals; and for the multiple delay case, Cao and Wu [5], Gopalsamy and He [11], van den Driessche and Zou [18], and Ye et al. [25] also attacked the global convergence of the network via Lyapunov functions/functionals and monotone dynamical systems theory.

By using a Lyapunov functional, Gopalsamy and He [11] obtained the following result.

Theorem 1.1. *Assume that* (H1) *and* (H2) *hold and* $\tau_{ij} \geq 0$. *If*

$$\max_{1\leq j\leq n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\} < 1, \tag{1.2}$$

then, for every input $J = (J_1, \dots, J_n)^{\mathrm{T}}$, *system* (1.1) *has a unique equilibrium* u^* *that is globally asymptotically stable, independent of the delays.*

This theorem provides an explicit criterion for global attractivity of (1.2). However, it has an obvious drawback: it neglects the signs of the connecting weights, and thus, does not distinguish the differences between excitatory ($a_{ij} > 0$) and inhibitory ($a_{ij} < 0$) connections. By using monotone dynamical systems theory and the so-called "exponential ordering" (see [16] for details), van den Driessche and Zou [18] showed that for a network with *non-inhibitory interconnections, inhibitory self-connections* (i.e., negative self-connections) can stabilize the network if the delays in inhibitory feedback channels are sufficiently small. More precisely, they established the following result in terms of the matrix measure:

$$\mu_1(A) \triangleq \max_{1\leq i\leq n} \left\{ a_{ii} + \sum_{j\neq i}^{n} |a_{ji}| \right\} \quad \text{for any } A \in R^{n\times n}.$$

Theorem 1.2 (van den Driessche and Zou [18, Theorems 3.1 and 3.4]). *Assume that* (H1) *and* (H2) *hold,* $\tau_{ij} \geq 0$ *and* $A = (a_{ij})$ *is an* $n \times n$ *irreducible matrix with* $a_{ij} \geq 0$ *for* $j \neq i$. *Assume also that either* $\mu_1(A) < 1$ *or* $\mu_1(\frac{1}{2}(A + A^{\mathrm{T}})) < 1$. *Then for every input* $J = (J_1, \dots, J_n)^{\mathrm{T}}$, *system* (1.1) *has a unique equilibrium, which is globally asymptotically stable provided that the diagonal delays* τ_{ii} *corresponding to negative* a_{ii} *are sufficiently small such that* $0 \leq \tau_{ii} \leq 1/(1 - ea_{ii})$.

Theorem 1.2 reveals the positive impact of *inhibitory self-connections*, that is, for a network with *non-inhibitory* interactions ($a_{ij} \geq 0$ for $i \neq j$), fast and relatively large inhibitory self-connections (small $\tau_{ii} \geq 0$ and $a_{ii} < 0$) ensure the desired global convergence to a unique equilibrium of the network. Our aim here is to show that the above assumptions of *non-inhibitory interactions* ($a_{ij} \geq 0$) and irreducibility of A are not crucial. In other words, we confirm the stabilization role of fast and large inhibitory self-connections for a delayed network with a general connection matrix. This is useful in applications as it allows for a much broader class of connection topologies in a network with delayed feedback to have a globally asymptotically stable equilibrium. Theoretically, such a confirmation is necessary for two reasons. Firstly, the work of Smith and Thieme [16] shows that excitatory effects tend to stabilize a network as the corresponding model generates a monotone semiflow, and the work of Baptistini and Taboas [1] establishes the existence of periodic solutions in a simple network of neurons involving inhibitory interactions and with large delay in signal transmission. Secondly, the work of Bélair et al. [3] shows that in a network of neurons with delayed feedback, frustration is a necessary condition for a network to generate nonlinear oscillation through Hopf bifurcation, and having inhibitory interaction is a minimal requirement for a network to be frustrated. Our approach is to embed a network into a larger system with only excitatory interactions, a technique previously employed to study the convergence problems for population dynamics by Cosner [9], and Wu and Zhao [22]. This approach allows us to apply the powerful theory of monotone dynamical systems to derive quite general sufficient conditions for global convergence.

2. Stabilization role of fast inhibitory self-connections

We show that Theorem 1.2 can be improved to include networks with more general interactions (i.e., without assuming $a_{ij} \geq 0$ for $j \neq i$ and without the irreducibility of A). More precisely, we prove the following theorem.

Theorem 2.1. *Assume that* (H1) *and* (H2) *hold, and* $\tau_{ij} \geq 0$. *If either* (*i*) $\mu_1(A) < 1$ *or* (*ii*)

$$\max_{1 \leq i \leq n} \left\{ a_{ii} + \sum_{j \neq i}^{n} \frac{|a_{ij}| + |a_{ji}|}{2} \right\} < 1,$$

then for every input $J = (J_1, \ldots, J_n)^{\mathrm{T}}$, *system* (1.1) *has a unique equilibrium, which is globally asymptotically stable provided that the diagonal delays* τ_{ii} *corresponding to negative* a_{ii} *are sufficiently small such that* $0 \leq \tau_{ii} \leq 1/(1 - ea_{ii})$.

Proof. For $j \neq i$, let $a_{ij}^{+} = \max\{a_{ij}, 0\}$ and $a_{ij}^{-} = \max\{-a_{ij}, 0\}$. Then both a_{ij}^{+} and a_{ij}^{-} are non-negative for $j \neq i$. Moreover $a_{ij} = a_{ij}^{+} - a_{ij}^{-}$ and $|a_{ij}| = a_{ij}^{+} + a_{ij}^{-}$ for $j \neq i$. Define $n \times n$ matrices $B = (b_{ij})$ and $C = (c_{ij})$ by

$$b_{ij} = \begin{cases} a_{ii} & \text{for } j = i, \\ a_{ij}^{+} + s & \text{for } j \neq i, \end{cases} \qquad c_{ij} = \begin{cases} 0 & \text{for } j = i, \\ a_{ij}^{-} + s & \text{for } j \neq i, \end{cases} \tag{2.1}$$

where $s > 0$ is a real number to be specified later. Now, since $a_{ij} = b_{ij} - c_{ij}$, (1.1) can be rewritten as

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -u_i(t) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t - \tau_{ij})) - \sum_{j=1}^{n} c_{ij} g_j(u_j(t - \tau_{ij})) + J_i. \tag{2.2}$$

Let $v_i = -u_i$, $i = 1, \ldots, n$. Then (2.2) (i.e. (1.1)) is embedded into the following $2n$-dimensional system:

$$\begin{aligned} \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} &= -u_i(t) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t - \tau_{ij})) + \sum_{j=1}^{n} c_{ij} f_j(v_j(t - \tau_{ij})) + J_i, \\ \frac{\mathrm{d}v_i(t)}{\mathrm{d}t} &= -v_i(t) + \sum_{j=1}^{n} c_{ij} g_j(u_j(t - \tau_{ij})) + \sum_{j=1}^{n} b_{ij} f_j(v_j(t - \tau_{ij})) - J_i, \quad i = 1, \ldots, n, \end{aligned} \tag{2.3}$$

where f_i is defined by $f_i(x) = -g_i(-x)$ for $i = 1, \dots, n$ and for $x \in R$, and thus f_i also satisfies (H1) and (H2). Further define $x_i(t)$, $i = 1, \dots, 2n$, by

$$x_i(t) = u_i(t), \qquad x_{n+i}(t) = v_i(t), \quad i = 1, \dots, n,$$

and define $h_i(x)$, $i = 1, \dots, 2n$ by

$$h_i(x) = g_i(x), \qquad h_{n+i}(x) = f_i(x), \quad x \in R, \; i = 1, \dots, n.$$

Then (2.3) can be written as

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + \sum_{j=1}^{2n} w_{ij} h_j(x_j(t - T_{ij})) + K_i, \quad i = 1, \dots, 2n, \tag{2.4}$$

where $K_i = J_i$ and $K_{n+i} = -J_i$ for $i = 1, \dots, n$, the $2n \times 2n$ matrix $W = (w_{ij})$ is given by

$$W = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \tag{2.5}$$

and each T_{ij}, $i, j = 1, \dots, 2n$, is given by

$$T_{ij} = T_{n+i,j} = T_{i,n+j} = T_{n+i,n+j} = \tau_{ij}, \quad i, j = 1, \dots, n. \tag{2.6}$$

By (2.1), it is obvious that $b_{ij} > 0$ and $c_{ij} > 0$ for $j \neq i$, and thus, the $2n \times 2n$ matrix W is irreducible. In order to apply Theorem 1.2 to (2.4), we need to compute $\mu_1(W)$ and $\mu_1(\frac{1}{2}(W + W^{\mathrm{T}}))$.

For any $k \in \{1, \dots, 2n\}$, there are two cases for computing $\mu_1(W)$:

Case 1 ($1 \leq k \leq n$). In this case,

$$w_{kk} + \sum_{j \neq k}^{2n} |w_{jk}| = w_{kk} + \sum_{j \neq k}^{n} |w_{jk}| + \sum_{j=1}^{n} |w_{n+j,k}| = b_{kk} + \sum_{j \neq k}^{n} b_{jk} + \sum_{j=1}^{n} c_{jk}.$$

From (2.1) it follows that

$$\begin{aligned} w_{kk} + \sum_{j \neq k}^{2n} |w_{jk}| &= b_{kk} + \sum_{j \neq k}^{n} (b_{jk} + c_{jk}) = a_{kk} + \sum_{j \neq k}^{n} [(a_{jk}^{+} + s) + (a_{jk}^{-} + s)] \\ &= a_{kk} + \sum_{j \neq k}^{n} [(a_{jk}^{+} + a_{jk}^{-}) + 2s] = a_{kk} + \sum_{j \neq k}^{n} |a_{jk}| + 2(n-1)s. \end{aligned} \tag{2.7}$$

Case 2 ($n + 1 \leq k \leq 2n$). Let $k = n + i$, where $1 \leq i \leq n$. Then,

$$w_{kk} + \sum_{j \neq k}^{2n} |w_{jk}| = w_{kk} + \sum_{j=1}^{n} |w_{jk}| + \sum_{j=n+1, j \neq k}^{2n} |w_{jk}| = b_{ii} + \sum_{j=1}^{n} c_{ji} + \sum_{j \neq i}^{n} b_{ji} = a_{ii} + \sum_{j \neq i}^{n} |a_{ji}| + 2(n-1)s. \tag{2.8}$$

Combining Cases 1 and 2,

$$\mu_1(W) = \max_{1 \leq i \leq n} \left\{ a_{ii} + \sum_{j \neq i}^{n} |a_{ji}| + 2(n-1)s \right\}. \tag{2.9}$$

Similarly, we can establish the following:

$$\mu_1\left(\frac{W+W^{\mathrm{T}}}{2}\right)=\max_{1\le i\le n}\left\{a_{ii}+\sum_{j\ne i}^{n}\frac{|a_{ij}|+|a_{ji}|}{2}+2(n-1)s\right\}. \tag{2.10}$$

Therefore, if either condition (i) or (ii) is satisfied, then (2.9) or (2.10) enable us to choose $s>0$ sufficiently small such that either $\mu_1(W)<1$ or $\mu_1(\frac{1}{2}(W+W^{\mathrm{T}}))<1$. For such an $s>0$, application of Theorem 1.2 to (2.4) ensures that (2.4) has a unique equilibrium x^*, which is globally asymptotically stable if the diagonal delays in (2.3) corresponding to negative w_{ii}, $i=1,\dots,2n$, are sufficiently small such that $0\le T_{ii}\le 1/(1-ew_{ii})$. Note that $w_{n+i,n+i}=w_{ii}=b_{ii}=a_{ii}$ and $T_{ii}=T_{n+i,n+i}=\tau_{ii}$ for $i=1,\dots,n$. This guarantees that every solution of (1.1) also converges to a unique equilibrium u^* of (1.1) provided that the diagonal delays $\tau_{ii}\ge 0$ corresponding to negative a_{ii} are sufficiently small such that $0\le\tau_{ii}\le 1/(1-ea_{ii})$. □

The result of Theorem 2.1 shows that, under certain restrictions, the network is globally convergent provided that the diagonal delays in the inhibitory self-connection channels are sufficiently small. However, Hopf bifurcation can occur as these delays are increased, as some numerical simulations in the following section suggest. Note that Theorem 2.1 imposes no restrictions on the sizes of the delays in the interconnection channels and the non-inhibitory self-connection channels.

3. Examples and numerical simulations

In this section, we consider some particular networks of two neurons and give supporting numerical simulations. Wei and Ruan [19] considered the network

$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}=-u_1(t)+2\tanh[u_2(t-\tau_{12})],\qquad \frac{\mathrm{d}u_2(t)}{\mathrm{d}t}=-u_2(t)-1.5\tanh[u_1(t-\tau_{21})] \tag{3.1}$$

and showed that when $\tau_{12}+\tau_{21}$ is increased to pass through the critical value 0.8, the origin loses its stability and Hopf bifurcation occurs. The bifurcation is supercritical and the periodic solution is orbitally asymptotically stable. Their numerical simulations at $\tau_{12}=0.325$ and $\tau_{21}=0.525$ (hence $\tau_{12}+\tau_{21}>0.8$) supported their conclusions ([19], Fig. 2). We add two inhibitory self-connections and consider the new network

$$\begin{aligned}\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}&=-u_1(t)-1.5\tanh[u_1(t-\tau_{11})]+2\tanh[u_2(t-\tau_{12})],\\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t}&=-u_2(t)-1.5\tanh[u_1(t-\tau_{21})]-0.6\tanh[u_2(t-\tau_{22})].\end{aligned} \tag{3.2}$$

By Theorem 2.1, this network converges globally to the unique equilibrium (the origin), provided $\tau_{11}<1/(1+1.5e)$ and $\tau_{22}<1/(1+0.6e)$ (but for arbitrary values of $\tau_{12}\ge 0$ and $\tau_{21}\ge 0$). This is illustrated by the numerical simulation shown in Fig. 1 in which $\tau_{12}=0.325$, $\tau_{21}=0.525$, $\tau_{11}=0.1$ and $\tau_{22}=0.2$. Note that Theorem 1.1 does not apply to (3.2).

Note that condition (i) in Theorem (2.1) is equivalent to

$$\sum_{j\ne i}^{n}|a_{ji}|<1-a_{ii},\quad i=1,\dots,n, \tag{3.3}$$

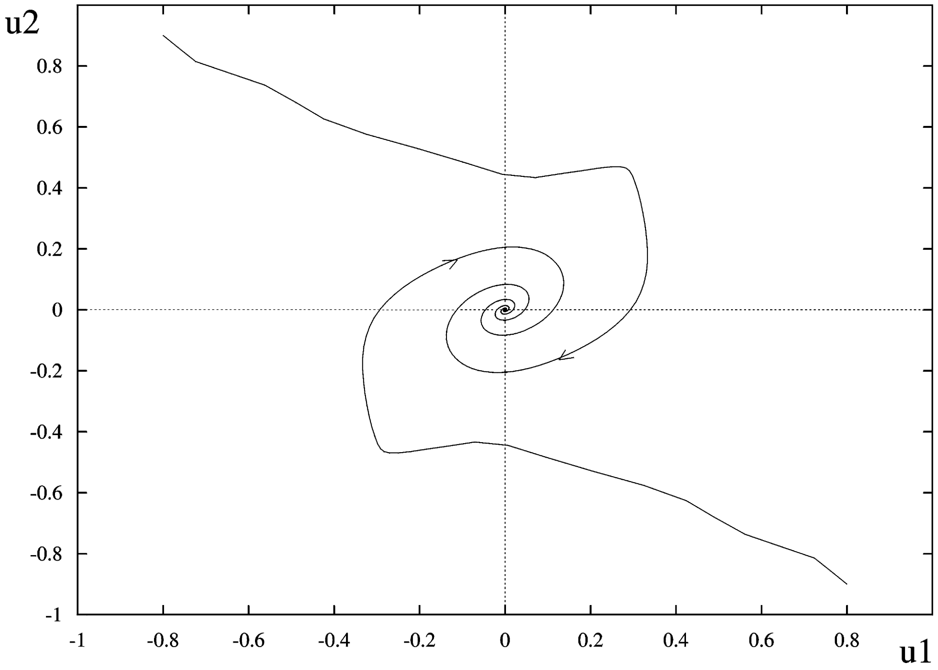

Fig. 1. Numerical simulation of solutions of (3.2) with $\tau_{12} = 0.325$, $\tau_{21} = 0.525$, $\tau_{11} = 0.1$, $\tau_{22} = 0.2$.

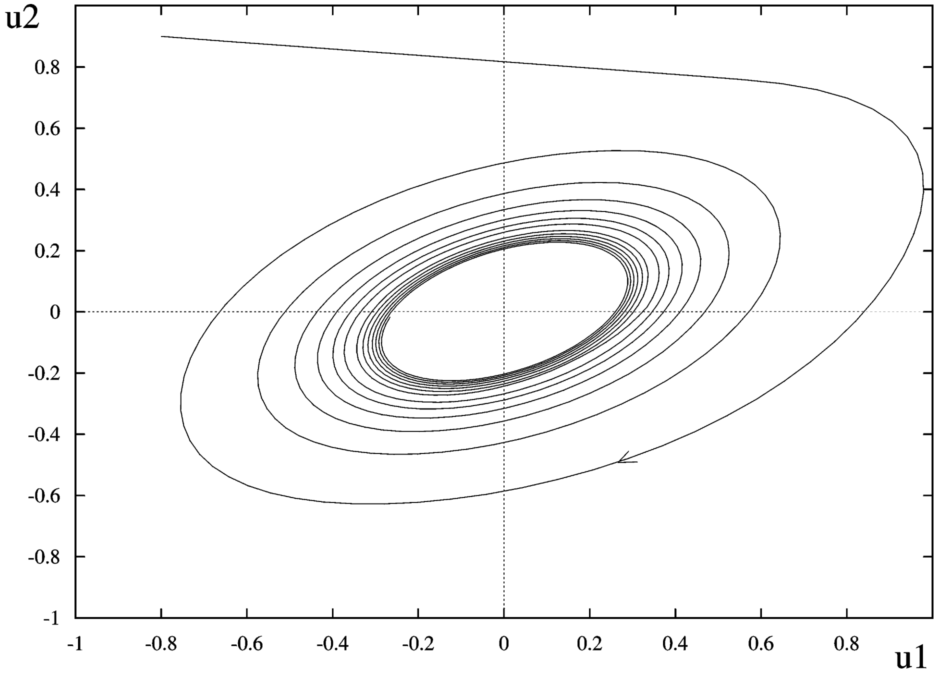

Fig. 2. Numerical simulation of solutions of (3.2) with τ_{12} and τ_{21} as for Fig. 1, but $\tau_{11} = \tau_{22} = 0.6$.

which represents a kind of “diagonal dominance” for the network (1.1) when a_{ii} are negative. Similarly, condition (ii) in Theorem (2.1) also gives another “diagonal dominance” condition

$$\sum_{j\neq i}^{n}\frac{|a_{ij}|+|a_{ji}|}{2} < 1-a_{ii}, \quad i=1,\dots,n. \tag{3.4}$$

It should be pointed out that the stabilization role of the inhibitory self-connections is not simply due to the diagonal domination (3.3) and (3.4), it has to be associated with the restrictions on delays in the self-connection channels. To see this, fix $\tau_{12}=0.325$ and $\tau_{21}=0.525$, but increase $\tau_{11}=\tau_{22}=\tau$. Numerical simulations show that when τ becomes large enough, the trivial solution of (3.2) loses its stability and sustained oscillations occur; see Fig. 2 for the numerical simulations in which $\tau=0.6>\max\{1/(1+1.5e), 1/(1+0.6e)\}$. The simulations were done by using XPPAUT [10].

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